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By a group of supervisors

THE MAIN BOOK

3rd PREP. 2024

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Maths

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Preface

Thanks to God who helped us to introduce one of our famous series **"El Moasser"** in mathematics.

We introduce this book to our colleagues.
We also introduce it to our students to help them study mathematics.

In fact, this book is the outcome of more than thirty years experience in the field of teaching mathematics.

This book will make students aware of all types of questions.

We would like to know your opinions about the book hoping that it will win your admiration.

We will be grateful if you send us your recommendations and your comments.

the authors

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2- Education, elementary.

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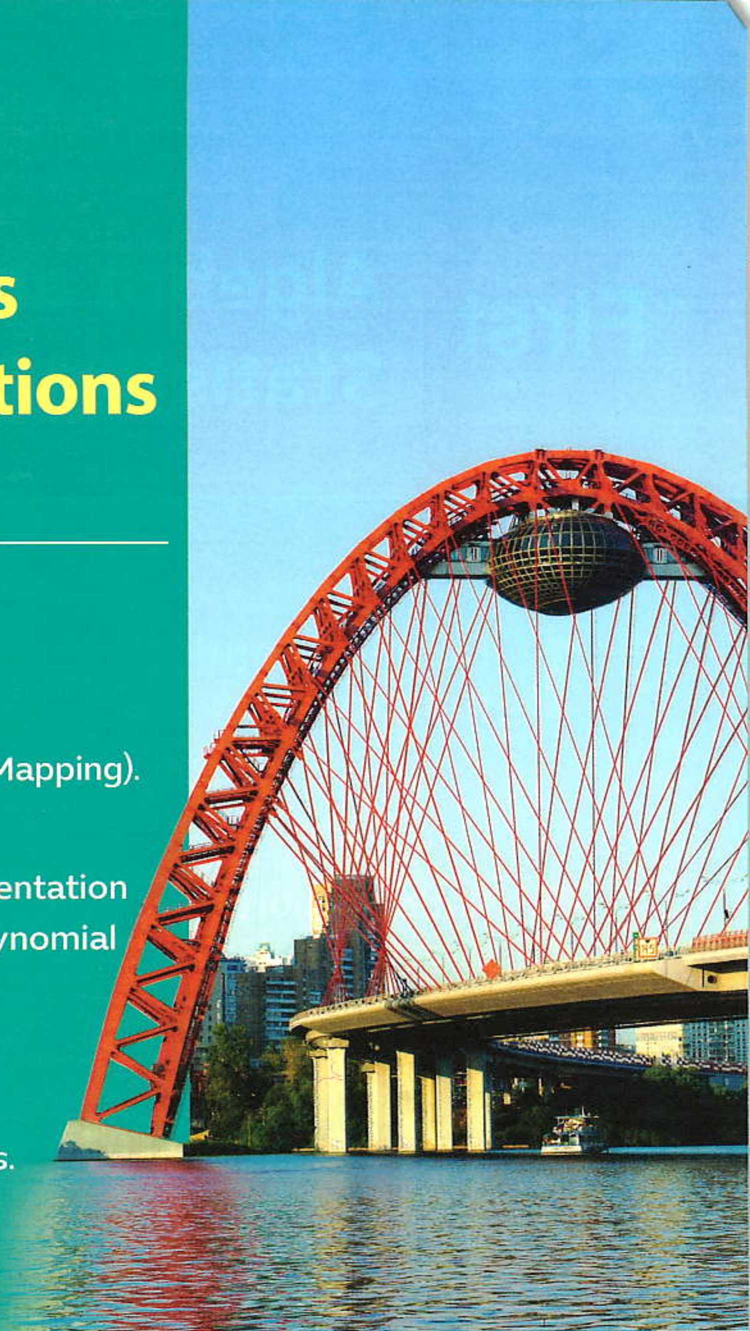
First | Algebra and Statistics

UNIT 1

Relations and functions



Unit Objectives : By the end of this unit, student should be able to :

- recognize the concept of the Cartesian product of two finite sets.
- represent the Cartesian product of two finite sets by the arrow diagram and the graphical (Cartesian) diagram.
- recognize the concept of the Cartesian product of two infinite sets.
- find the Cartesian product of two intervals.
- recognize the concept of the relation from a set to another one.
- recognize whether the relation is a function or not.
- represent the function by the arrow diagram and the graphical (Cartesian) diagram.
- recognize the domain , the codomain and the range of the function.
- express the function symbolically.
- search the degree of the polynomial function.
- represent the linear function graphically.
- recognize the constant function and represent it graphically.
- represent graphically the quadratic function.
- find the vertex of the curve of the quadratic function.
- find the maximum or the minimum value of the quadratic function.
- find the equation of the axis of symmetry of the quadratic function.

Use
your smart phone or tablet to scan the **QR code** and enjoy watching videos

Lesson 1

Cartesian product

In this lesson , we shall know the concept of the Cartesian product and how to find it and how to represent it graphically.

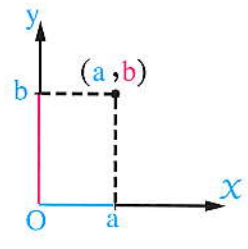
Before dealing with this subject , we shall remember together what we had studied about the ordered pair.

The ordered pair

(a , b) is called an ordered pair

- a is called the first projection
- b is called the second projection

and the ordered pair (a , b) could be represented by a point as shown in the opposite figure.

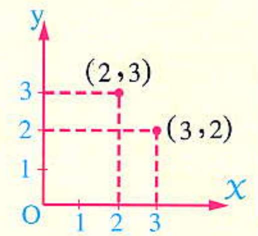


Remarks

- If $a \neq b$, then $(a , b) \neq (b , a)$

  For example: $(2 , 3) \neq (3 , 2)$

  and when representing them graphically as shown in the opposite figure , we find that they are represented by two different points.



- The ordered pair is not a set. i.e. $(a , b) \neq \{a , b\}$

- (a , a) is an ordered pair , while in the sets , we don't write $\{a , a\}$, but we write $\{a\}$ without repeating the element a
- There is an empty set of elements and denoted by the symbol $\varnothing$, but there is not an empty ordered pair.

The equality of two ordered pairs

If (a , b) = (x , y) , then a = x , b = y

For example:

- If (a , b) = (3 , – 4) , then a = 3 , b = – 4
- If (x , 2) = (– 5 , y) , then x = – 5 , y = 2

Example 1 **Choose the correct answer from the given ones :**

1 If $(3 , 8) = \left(3 , \sqrt{y}\right)$, then $\sqrt[3]{y}$ = ...

(a) – 4 (b) 4 (c) 8 (d) 64

2 If $(32 , x + y) = (y^5 , 2)$, then x = ...

(a) 0 (b) 2 (c) 4 (d) 5

3 If $(2^{x-1} , -3) = (1 , y)$, then $2x - y$ = ...

(a) – 3 (b) – 1 (c) 3 (d) 5

4 If $(x^2 - 1 , 4) = (48 , 2y)$, then xy = ...

(a) – 7 (b) 7 (c) 14 (d) ± 14

Solution

1 **(b) The reason :** $\because (3 , 8) = \left(3 , \sqrt{y}\right)$ $\therefore \sqrt{y} = 8$

$\therefore y = 8^2 = 64$ $\therefore \sqrt[3]{y} = \sqrt[3]{64} = 4$

2 **(a) The reason :** $\because (32 , x + y) = (y^5 , 2)$

$\therefore y^5 = 32$ $\therefore y = 2$ «because $2^5 = 32$»

, $x + y = 2$ substituting by y = 2 $\therefore x + 2 = 2$

$\therefore x = 0$

3 **(d) The reason :** $\because (2^{x-1} , -3) = (1 , y)$ $\therefore y = -3$

, $2^{x-1} = 1$, then $x - 1 = 0$ $\therefore x = 1$

$\therefore 2x - y = 2 \times 1 - (-3) = 2 + 3 = 5$

4 **(d) The reason :** $\because (x^2 - 1 , 4) = (48 , 2y)$ $\therefore x^2 - 1 = 48$

$\therefore x^2 = 49$

$\therefore x = \pm\sqrt{49} = \pm 7$, $2y = 4$ $\therefore y = \frac{4}{2} = 2$

$\therefore xy = \pm 7 \times 2 = \pm 14$

TRY by yourself 1 ***Find the values of x and y in each of the following :***

1. $(x + 1 , y^2) = (3 , 9)$
2. $(x^3 - 5 , 8) = (3 , 3y - 7)$
3. $(x^2 - 2 , 2y) = \left(y , \sqrt[3]{64}\right)$

The Cartesian product of two finite sets

For any two finite and non empty sets X and Y , we get :

The Cartesian product of the set X by the set Y and it is denoted by $X \times Y$ is the set of all ordered pairs whose first projection of each of them belongs to X and the second projection of each of them belongs to Y

i.e. $X \times Y = \{(a , b) : a \in X , b \in Y\}$

For example :

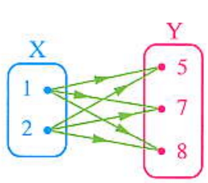
1. If $X = \{1 , 2\}$, $Y = \{5 , 7 , 8\}$, then :

$$X \times Y = \{1 , 2\} \times \{5 , 7 , 8\}$$

$$= \{(1 , 5) , (1 , 7) , (1 , 8) , (2 , 5) , (2 , 7) , (2 , 8)\}$$

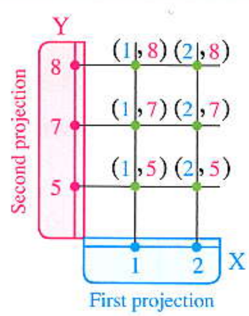
• We can represent $X \times Y$ by two ways as follows :

1st way : The arrow diagram



Where we draw an arrow going from each element representing the first projection (the elements of the set X) to each element representing the second projection (the elements of the set Y)

2nd way : The graphical (Cartesian) diagram

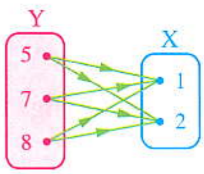


Where the elements of the set X are represented horizontally and the elements of the set Y are represented vertically and the points of intersection of the horizontal and vertical lines represent the Cartesian product of $X \times Y$

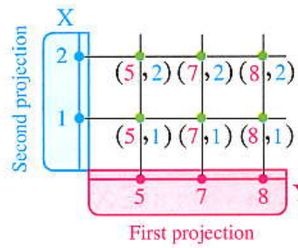
2 If $X = \{1, 2\}$, $Y = \{5, 7, 8\}$, then :

$$Y \times X = \{5, 7, 8\} \times \{1, 2\}$$
$$= \{(5, 1), (5, 2), (7, 1), (7, 2), (8, 1), (8, 2)\}$$

- Similarly , we can represent $Y \times X$ by two ways as follows :



The arrow diagram



The Cartesian diagram

The Cartesian product of a set by itself

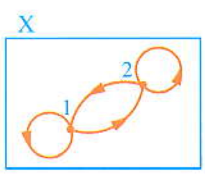
The Cartesian product of the set X by itself and we denote it by $X \times X$ or by X^2 (it is read X two) is the set of all ordered pairs whose first projections and second projections belong both to X

i.e. $X \times X = \{(a, b) : a \in X , b \in X\}$

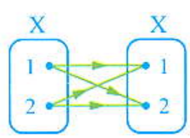
For example: If $X = \{1, 2\}$, then :

$$X \times X = \{1, 2\} \times \{1, 2\} = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$$

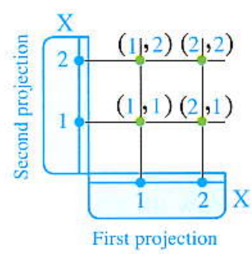
- We can represent $X \times X$ by two ways as follows :



or



The arrow diagram



The Cartesian diagram

Notice that : The figure  is called a loop to show that the arrow goes from the point and returns to the same point.

! Remarks

- For any two finite and non empty sets X and Y , then $X \times Y \neq Y \times X$ where $X \neq Y$
- For any set X , then $X \times \varnothing = \varnothing \times X = \varnothing$ where $\varnothing$ is the empty set.
- If $(a , b) \in X \times Y$, then $a \in X$, $b \in Y$

For example: If $(3 , 5) \in X \times Y$, then $3 \in X$, $5 \in Y$

Example 2 If $X = \{2 , 3 , 4\}$ and $Y = \{a , b\}$, **find each of :**

1 $X \times Y$ 2 $Y \times X$ 3 $X \times X$ 4 Y^2

Solution

1 $X \times Y = \{(2 , a) , (2 , b) , (3 , a) , (3 , b) , (4 , a) , (4 , b)\}$

2 $Y \times X = \{(a , 2) , (a , 3) , (a , 4) , (b , 2) , (b , 3) , (b , 4)\}$

3 $X \times X = \{(2 , 2) , (2 , 3) , (2 , 4) , (3 , 2) , (3 , 3) , (3 , 4) , (4 , 2) , (4 , 3) , (4 , 4)\}$

4 $Y^2 = \{(a , a) , (a , b) , (b , a) , (b , b)\}$

TRY by yourself 2 If $X = \{3 , 4 , 5\}$ and $Y = \{5 , 6\}$, ***find each of the following :***

1 $Y \times X$ and represent it by an arrow diagram

2 X^2 and represent it by a Cartesian diagram

The number of the elements of the Cartesian product

If we denote the number of elements of the set X by n (X) and the number of elements of the set Y by n (Y) , then the number of elements of the Cartesian product $X \times Y$ is denoted by $n (X \times Y)$, and :

- $n (X \times Y) = n (Y \times X) = n (X) \times n (Y)$
- $n (X \times X) = n (X) \times n (X) = [n (X)]^2$
- $n (X \times \varnothing) = n (X) \times n (\varnothing)$
  $= 0$ [Because $n (\varnothing) = 0$]

Notice that :

If X , Y are two finite and non empty sets , $X \neq Y$, then $X \times Y \neq Y \times X$, but $n (X \times Y) = n (Y \times X)$

For example :

If $X = \{2 , -1 , 0\}$ and $Y = \{5 , -7\}$, then $n (X) = 3$, $n (Y) = 2$, then :

- $n (X \times Y) = 3 \times 2 = 6$
- $n (Y \times X) = 2 \times 3 = 6$
- $n (X^2) = 3^2 = 9$
- $n (Y^2) = 2^2 = 4$

Find the previous Cartesian products and verify the number of their elements.

Example 3 **Choose the correct answer from the given ones :**

1 If $X = \{0 , 2\}$, $n(Y) = 5$, then $n(X \times Y) =$ …………

(a) 2 (b) 5 (c) 7 (d) 10

2 If $n(Y) = 4$, $n(X \times Y) = 8$, then $n(X) =$ …………

(a) 2 (b) 4 (c) 8 (d) 32

3 If $n(X^2) = 9$, $n(Y^2) = 16$, then $n(Y \times X) =$ …………

(a) 7 (b) 12 (c) 36 (d) 144

Solution

1 **(d) The reason :** $\because n(X) = 2$, $n(Y) = 5$

$\therefore n(X \times Y) = 2 \times 5 = 10$

2 **(a) The reason :** $n(X) = \frac{n(X \times Y)}{n(Y)} = \frac{8}{4} = 2$

3 **(b) The reason :** $\because n(X^2) = 9$ $\therefore n(X) = \sqrt{9} = 3$

, $\because n(Y^2) = 16$ $\therefore n(Y) = \sqrt{16} = 4$

$\therefore n(Y \times X) = 4 \times 3 = 12$

TRY by yourself 3 ***Choose the correct answer from the given ones :***

1 If $n(X) = 3$, $n(X \times Y) = 12$, then $n(Y) =$ …………

(a) 4 (b) 9 (c) 15 (d) 36

2 If $Y = \{-1 , 0 , 1\}$, $n(X \times Y) = 15$, then $n(Y^2) =$ …………

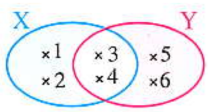
(a) 5 (b) 9 (c) 15 (d) 25

3 If $n(X^2) = 4$, $n(X \times Y) = 4$, then $n(Y^2) =$ …………

(a) 1 (b) 2 (c) 4 (d) 16

Remember the operations on sets

If $X = \{1 , 2 , 3 , 4\}$, $Y = \{3 , 4 , 5 , 6\}$, then :



- $X \cap Y$ = the set of elements which are common in X and Y = $\{3 , 4\}$
- $X \cup Y$ = the set of all elements in X or Y without repeating = $\{1 , 2 , 3 , 4 , 5 , 6\}$
- $X - Y$ = the set of elements which are in X and not in Y = $\{1 , 2\}$
- $Y - X$ = the set of elements which are in Y and not in X = $\{5 , 6\}$

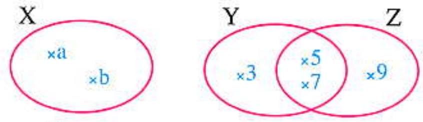
Example 4

If $X = \{a , b\}$, $Y = \{3 , 5 , 7\}$, $Z = \{5 , 7 , 9\}$

, represent the sets X , Y and Z by Venn diagram , then find :

1. $X \times (Y \cup Z)$, $(X \times Y) \cup (X \times Z)$
2. $X \times (Y \cap Z)$, $(X \times Y) \cap (X \times Z)$
3. $X \times (Z - Y)$, $(X \times Z) - (X \times Y)$

Solution



1 $\because Y \cup Z = \{3 , 5 , 7 , 9\}$

$\therefore X \times (Y \cup Z) = \{a , b\} \times \{3 , 5 , 7 , 9\}$

$= \{(a , 3) , (a , 5) , (a , 7) , (a , 9) , (b , 3) , (b , 5) , (b , 7) , (b , 9)\}$

$, X \times Y = \{a , b\} \times \{3 , 5 , 7\}$

$= \{(a , 3) , (a , 5) , (a , 7) , (b , 3) , (b , 5) , (b , 7)\}$ (1)

$, X \times Z = \{a , b\} \times \{5 , 7 , 9\}$

$= \{(a , 5) , (a , 7) , (a , 9) , (b , 5) , (b , 7) , (b , 9)\}$ (2)

From (1) and (2) :

$\therefore (X \times Y) \cup (X \times Z) =$

$\{(a , 3) , (a , 5) , (a , 7) , (a , 9) , (b , 3) , (b , 5) , (b , 7) , (b , 9)\}$

2 $\because Y \cap Z = \{5 , 7\}$

$\therefore X \times (Y \cap Z) = \{a , b\} \times \{5 , 7\}$

$= \{(a , 5) , (a , 7) , (b , 5) , (b , 7)\}$

From (1) and (2) :

$\therefore (X \times Y) \cap (X \times Z) = \{(a , 5) , (a , 7) , (b , 5) , (b , 7)\}$

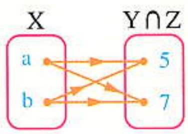
3 $\because Z - Y = \{9\}$

$\therefore X \times (Z - Y) = \{a , b\} \times \{9\} = \{(a , 9) , (b , 9)\}$

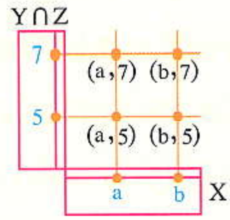
From (1) and (2) : $\therefore (X \times Z) - (X \times Y) = \{(a , 9) , (b , 9)\}$

Remark

In the previous example , we can represent $X \times (Y \cap Z)$ by an arrow diagram and a Cartesian diagram as follows :



The arrow diagram



The Cartesian diagram

TRY by yourself 4

If $X = \{2, 3\}$, $Y = \{1, 3, 5\}$, $Z = \{2\}$

, represent each of X , Y and Z by Venn diagram , then find :

1 $Z \times (X \cap Y)$ 2 $(Z \times X) \cup (Z \times Y)$

The Cartesian product of two infinite sets

- We know that if X is a finite set (having n elements) , then the Cartesian product $X \times X$ is also a finite set (having n^2 elements).

  For example: If $n(X) = 3$, then $n(X \times X) = 9$

- But if X is an infinite set , then $X \times X$ is an infinite set also

  As examples for that :

  $\mathbb{N} \times \mathbb{N} = \{(x, y) : x \in \mathbb{N}, y \in \mathbb{N}\}$, $\mathbb{Z} \times \mathbb{Z} = \{(x, y) : x \in \mathbb{Z}, y \in \mathbb{Z}\}$,

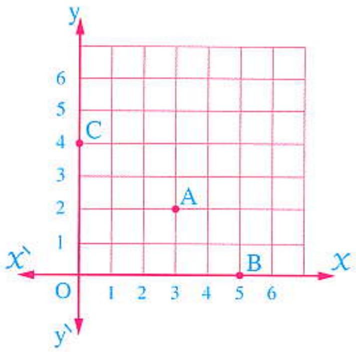
  $\mathbb{Q} \times \mathbb{Q} = \{(x, y) : x \in \mathbb{Q}, y \in \mathbb{Q}\}$, $\mathbb{R} \times \mathbb{R} = \{(x, y) : x \in \mathbb{R}, y \in \mathbb{R}\}$

- We know that if X is a finite set , we represent the Cartesian product $X \times X$ graphically by a finite number of points.

- But if X is an infinite set , then the Cartesian product $X \times X$ is represented graphically by an infinite number of points.

  The following is the graphical representation of each of : $\mathbb{N} \times \mathbb{N}$, $\mathbb{Z} \times \mathbb{Z}$, $\mathbb{R} \times \mathbb{R}$:

First Representing the Cartesian product $\mathbb{N} \times \mathbb{N}$ $(\mathbb{N}^2)$

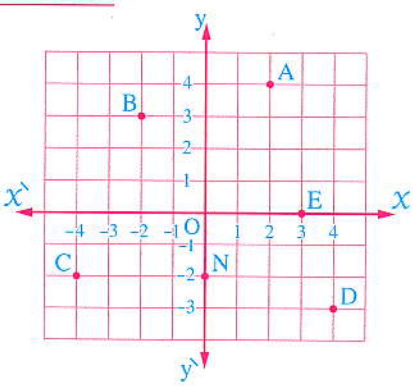


- Represent the natural numbers on two perpendicular straight lines, one of them $\overleftrightarrow{XX}$ is horizontal and the other $\overleftrightarrow{yy}$ is vertical , where they intersect at the point which represents the number zero on each of them **i.e.** O = (0 , 0)
- The opposite figure shows a small part of the perpendicular graphical net of the Cartesian product $\mathbb{N} \times \mathbb{N}$ which consists of the vertical and the horizontal straight lines that pass through the points which represent the natural numbers on each of $\overleftrightarrow{XX}$ and $\overleftrightarrow{yy}$
- And each point of the points of this net represents an ordered pair of the Cartesian product $\mathbb{N} \times \mathbb{N}$

For example :

- The point A represents the ordered pair (3 , 2)
- The point B represents the ordered pair (5 , 0)
- The point C represents the ordered pair (0 , 4)
- The point O represents the ordered pair (0 , 0)

Second Representing the Cartesian product $\mathbb{Z} \times \mathbb{Z}$ $(\mathbb{Z}^2)$



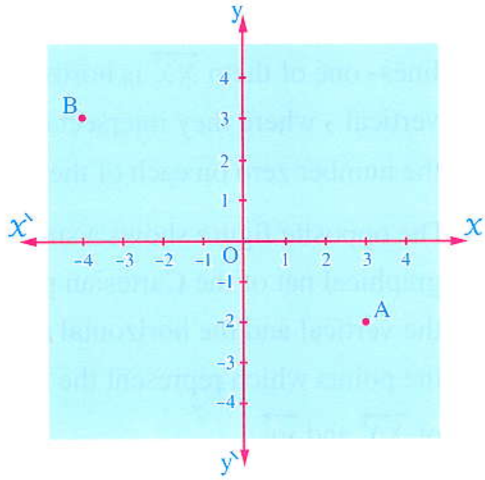
- Represent the integers on each of $\overleftrightarrow{XX}$ and $\overleftrightarrow{yy}$ which are intersecting at O (0 , 0)
- The opposite figure shows a small part of the perpendicular graphical net of the Cartesian product $\mathbb{Z} \times \mathbb{Z}$
- And each point of its points represents an ordered pair of the Cartesian product $\mathbb{Z} \times \mathbb{Z}$

For example:

- The point A represents the ordered pair (2 , 4)
- The point B represents the ordered pair (– 2 , 3)
- The point C represents the ordered pair (– 4 , – 2)
- The point D represents the ordered pair (4 , – 3)
- The point E represents the ordered pair (3 , 0)
- The point N represents the ordered pair (0 , – 2)

Third Representing the Cartesian product $\mathbb{R} \times \mathbb{R}$ $(\mathbb{R}^2)$

- The perpendicular graphical net of the Cartesian product $\mathbb{R} \times \mathbb{R}$ is an infinite extended surface from all sides and the opposite figure shows a small part of this region.
- Each point of this region represents an ordered pair of the Cartesian product $\mathbb{R} \times \mathbb{R}$

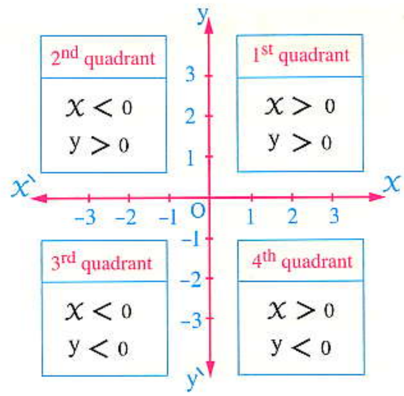


For example:

- The point A represents the ordered pair $(3 , -2)$
- The point B represents the ordered pair $(-4 , 3)$

Remarks

1. The horizontal straight line $\overleftrightarrow{XX'}$ is called X-axis or the horizontal axis and the vertical straight line $\overleftrightarrow{yy'}$ is called y-axis or the vertical axis.
2. The point of intersection of the two axes $\overleftrightarrow{XX'}$ and $\overleftrightarrow{yy'}$ is called the origin point.
3. If the point A represents the ordered pair (X , y) in the Cartesian product $\mathbb{R} \times \mathbb{R}$, then :
   - The first projection X is called the X-coordinate of the point A
   - The second projection y is called the y-coordinate of the point A
4. The two axes $\overleftrightarrow{XX'}$ and $\overleftrightarrow{yy'}$ divide the plane into four quadrants as shown in the opposite figure and we can determine the quadrant in which any point lies by knowing the signs of its two coordinates.



5. If the X-coordinate of the point = 0 , then the point lies on y-axis.
6. If the y-coordinate of the point = 0 , then the point lies on X-axis.

Example 5

Choose the correct answer from the given ones :

1 The point $(4 , -3)$ lies on the .... quadrant.

(a) first (b) second (c) third (d) fourth

2 Which of the following points lies on the third quadrant ?

(a) $(2 , 5)$ (b) $(2 , -5)$ (c) $(-2 , 5)$ (d) $(-2 , -5)$

3 If the point (a , 3 – a) lies on the X-axis , then a = ...

(a) – 3 (b) 0 (c) 3 (d) 5

4 If b < 2 , then the point (b – 2 , 4) lies on the .... quadrant.

(a) first (b) second (c) third (d) fourth

5 If the point (X – 3 , 4 – X) where $X \in \mathbb{Z}$ lies on the fourth quadrant , then X = ...

(a) 2 (b) 3 (c) 4 (d) 5

Solution

1 **(d) The reason :** Because the X-coordinate is positive and the y-coordinate is negative.

2 **(d) The reason :** Because the X-coordinate and the y-coordinate of all the points on the third quadrant are negative.

3 **(c) The reason :** $\because$ (a , 3 – a) $\in \overleftrightarrow{XX}$

$\therefore$ 3 – a = 0 $\therefore$ a = 3

4 **(b) The reason :** $\because$ b < 2

$\therefore$ The X-coordinate of the point (b – 2 , 4) is negative and its y-coordinate is positive.

$\therefore$ (b – 2 , 4) lies on the second quadrant.

5 **(d) The reason :** Because at X = 5 , then (X – 3 , 4 – X) = (2 , – 1)

i.e. The X-coordinate is positive and the y-coordinate is negative.

TRY by yourself 5

Choose the correct answer from the given ones :

1 The point (– 2 , – 7) lies on the .... quadrant.

(a) first (b) second (c) third (d) fourth

2 If the point (b – 5 , b) lies on the y-axis , then b = ...

(a) – 5 (b) 0 (c) 1 (d) 5

3 If $\left(X - 2 , \sqrt{9}\right) = (-3 , y)$, then the point (y , X) lies on the .... quadrant.

(a) first (b) second (c) third (d) fourth

4 The point (X^2 , y^2) where $X \neq 0$, $y \neq 0$ lies on the .... quadrant.

(a) first (b) second (c) third (d) fourth

The Cartesian product of two intervals

We studied that the interval is a subset of the set of the real numbers ($\mathbb{R}$) and then the Cartesian product of two intervals is a subset of the Cartesian product $\mathbb{R} \times \mathbb{R}$ and we can explain that in the following example.

Example 6 If $X = [0 , 3]$, $Y = [1 , 3]$

, represent graphically using the perpendicular graphical net of the Cartesian product $\mathbb{R} \times \mathbb{R}$ the region which represents each of :

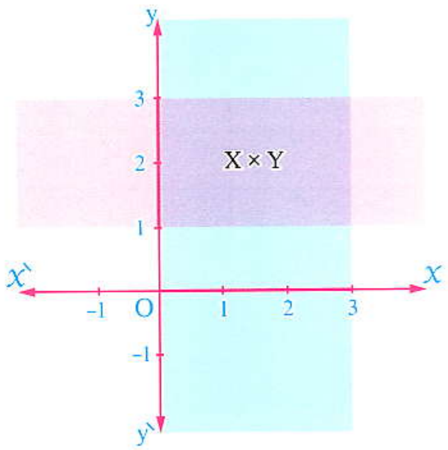
1 $X \times Y$ 2 $X \times X$ 3 $Y \times Y$

, then show , in each case , which of the following points belongs to the previous Cartesian products : (2 , 2) , (1 , 0) , (0 , 3)

Solution

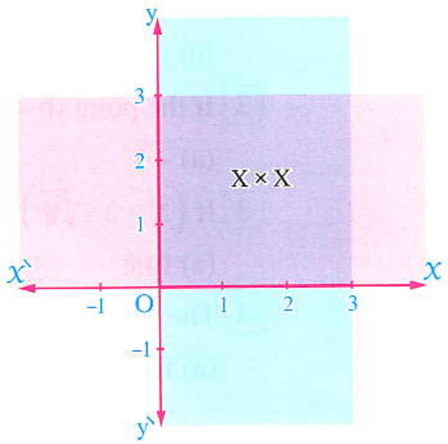
1 **To represent $X \times Y$ graphically , do as follows :**

- Represent the interval X on x-axis
- Represent the interval Y on y-axis
- The intersection region of the two colours represents $X \times Y$
- $(2 , 2) \in X \times Y$ because it belongs to the region which represents $X \times Y$
- $(1 , 0) \notin X \times Y$ because it lies outside the region which represents $X \times Y$
- $(0 , 3) \in X \times Y$



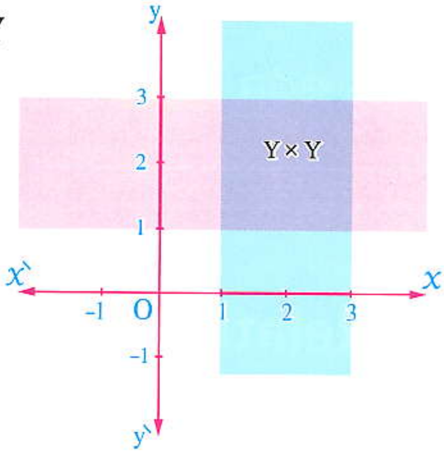
2 **To represent $X \times X$ graphically , do as follows :**

- Represent the interval X one time on x-axis and another time on y-axis.
- The intersection region of the two colours represents $X \times X$
- $(2 , 2) \in X \times X$, $(1 , 0) \in X \times X$ and $(0 , 3) \in X \times X$



3 **Similarly , you can represent Y × Y as shown in the opposite figure :**

- (2 , 2) ∈ Y × Y , (1 , 0) ∉ Y × Y and (0 , 3) ∉ Y × Y



Lesson 2

Relation - Function (Mapping)

First The relation

The relation from set X to set Y is a connection that connects some or all the elements of set X with some or all the elements of set Y and it is denoted by "R"

- The relation R from X to Y is a set of ordered pairs whose first projection belongs to X and its second projection belongs to Y and the first projection is connected with the second projection by this relation.

  If $(a , b) \in R$ where $a \in X , b \in Y$

  So , we express this as "a R b"

- The relation R from set X to set Y is a subset of the Cartesian product $X \times Y$

  i.e. $R \subset X \times Y$

- The relation can be expressed by an arrow diagram or a Cartesian diagram (graphical).

Example 1 If $X = \{2, 5\}$, $Y = \{1, 4, 7\}$ and R is a relation from X to Y where "a R b" means "a < b" for every $a \in X$, $b \in Y$, state the relation R and represent it by an arrow diagram and by a Cartesian diagram.

Solution

$\because$ 2 is not less than 1 $\quad\therefore (2, 1) \notin R$

$\because 2 < 4 \quad\therefore (2, 4) \in R$

$\because 2 < 7 \quad\therefore (2, 7) \in R$

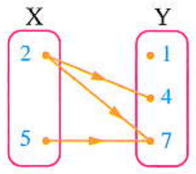
$\because$ 5 is not less than 1 $\quad\therefore (5, 1) \notin R$

$\because$ 5 is not less than 4 $\quad\therefore (5, 4) \notin R$

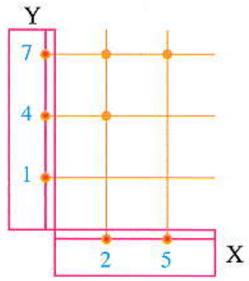
$\because 5 < 7 \quad\therefore (5, 7) \in R$

$\therefore$ The relation $R = \{(2, 4), (2, 7), (5, 7)\}$

The following figures represent the arrow diagram and the Cartesian diagram of this relation :



The arrow diagram



The Cartesian diagram

TRY by yourself 1

If $X = \{1, 2, 3\}$, $Y = \{3, 4, 5, 6\}$ and R is a relation from X to Y where "a R b" means "a + b = 6" for every $a \in X$ and $b \in Y$, state the relation R and represent it by an arrow diagram.

! Remark

If R is a relation from X to X, then : R is a relation on X and the relation $R \subset X \times X$

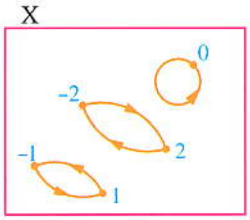
Example 2

If X = {– 2 , – 1 , 0 , 1 , 2 } and R is a relation on X where "a R b" means "a is the additive inverse of the number b" for every a ∈ X and b ∈ X , state R , then represent it by an arrow diagram and a Cartesian diagram.

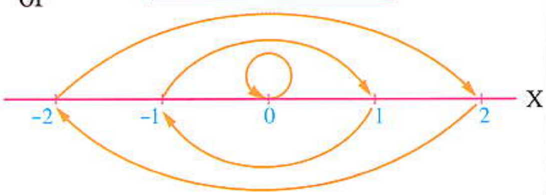
Solution

R = {(– 2 , 2) , (– 1 , 1) , (0 , 0) , (1 , – 1) , (2 , – 2)}

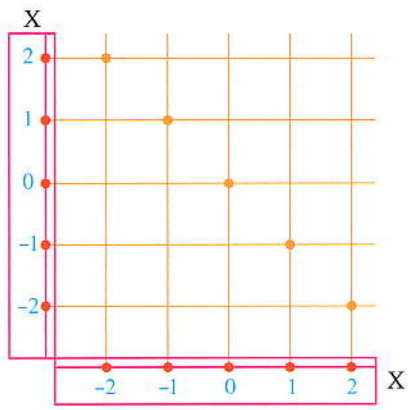
• **The arrow diagram :**



or

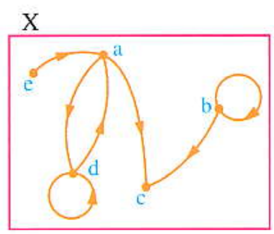


• **The Cartesian diagram :**



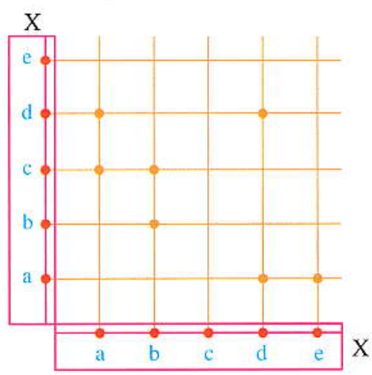
Example 3

If the opposite arrow diagram represents the relation R on X , state R , then represent it by a Cartesian diagram.



Solution

R = {(a , c) , (a , d) , (b , b) , (b , c) , (d , d) , (d , a) , (e , a)}



TRY by yourself 2

If X = {1 , 2 , 4} and R is a relation on X where "a R b" means "a is twice b" for every a ∈ X and b ∈ X , state R and represent it by a Cartesian diagram.

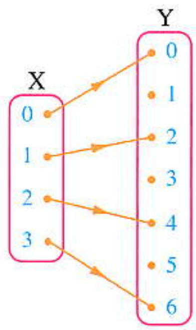
Second Function (Mapping)

Introductory example

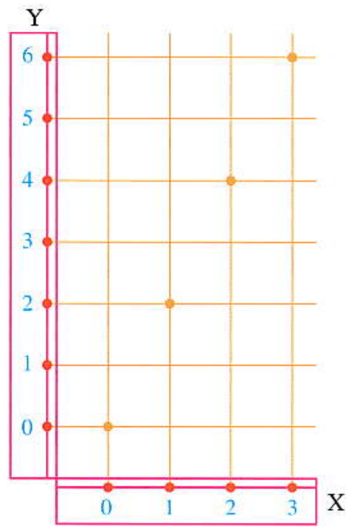
If X = $\{0, 1, 2, 3\}$, Y = $\{0, 1, 2, 3, 4, 5, 6\}$ and R is a relation from X to Y where "a R b" means "a = $\frac{1}{2}$ b" for each a $\in$ X , b $\in$ Y , write R and represent it by an arrow diagram and a Cartesian diagram.

Solution

R = $\{(0, 0), (1, 2), (2, 4), (3, 6)\}$



The arrow diagram

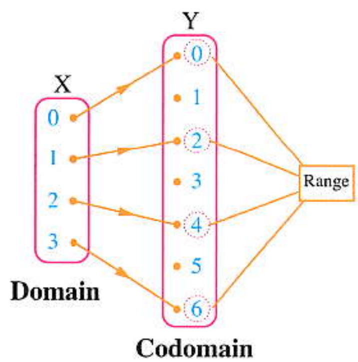


The Cartesian diagram

In the previous relation , we notice that :

Each element of the set X has been connected with **one and only one element** of the elements of the set Y

Such as , this relation is called **a function or (mapping).**



- The set X = $\{0, 1, 2, 3\}$
  is called "the domain of the function".
- The set Y = $\{0, 1, 2, 3, 4, 5, 6\}$
  is called "the codomain of the function".
- The set $\{0, 2, 4, 6\}$
  is called "the range of the function" and it is a subset from the codomain of the function.

Generally

A relation from X to Y is said to be a function if one of the following cases is satisfied :

1. In the relation , **each element** of the set X appears **only once** as a first projection in one of the ordered pairs of **the relation**.
2. In the arrow diagram which represents the relation , **each element** of X has **one and only one arrow** going out of it to one element of Y
3. In the Cartesian diagram which represents the relation , **each vertical line** has **one and only one point** lying on it of the points which represent the relation.

Example 4 If $X = \{1 , 2 , 3 , 4\}$, $Y = \{1 , 3 , 5 , 7\}$

, show which of the following relations represents a function from X to Y and if it is a function , mention its range :

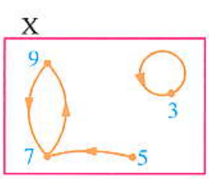
- $R_1 = \{(2 , 3) , (1 , 1) , (3 , 5) , (3 , 7) , (4 , 3)\}$
- $R_2 = \{(1 , 7) , (2 , 5) , (4 , 1)\}$
- $R_3 = \{(2 , 3) , (3 , 3) , (1 , 5) , (4 , 7)\}$

Solution

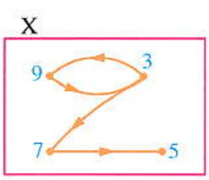
- R_1 is not a function because the element $3 \in X$ appears as a first projection twice in two ordered pairs of the relation (3 , 5) and (3 , 7)
- R_2 is not a function because the element $3 \in X$ does not appear as a first projection in any ordered pair of the relation.
- R_3 is a function because each element of X appeared only once as a first projection in an ordered pair of the relation , the range of R_3 is $\{3 , 5 , 7\}$

Example 5 If $X = \{3 , 5 , 7 , 9\}$

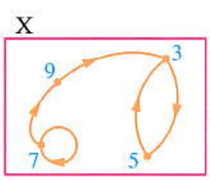
, show which of the following arrow diagrams represents a function on X (i.e. from X to X) and if it is a function , mention its range :



F_1



F_2



F_3

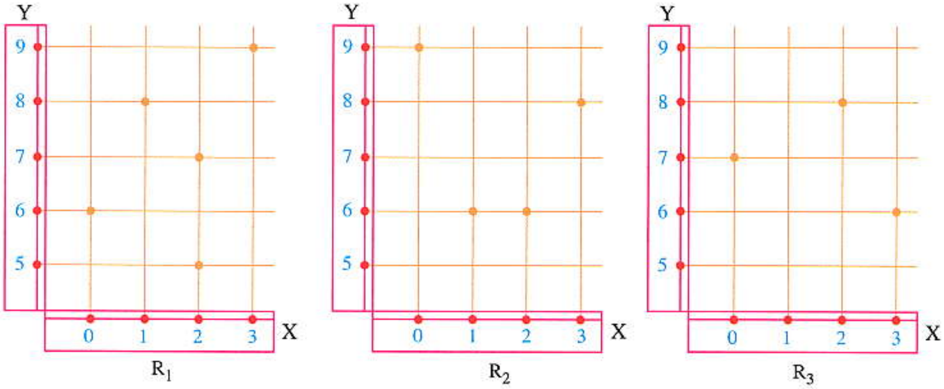
Solution

- F_1 is a function because each element of X has only one arrow going out of it to one element of X , the range of the function F_1 is $\{3, 7, 9\}$
- F_2 is not a function because for the element $5 \in X$ there are no arrows going out of it or because the element $3 \in X$ has two arrows going out of it.
- F_3 is not a function because the element $7 \in X$ has two arrows going out of it.

Example 6

If $X = \{0, 1, 2, 3\}$, $Y = \{5, 6, 7, 8, 9\}$

, show which of the following Cartesian diagrams represents a function from X to Y and if it is a function , mention its range :



Solution

- R_1 is not a function because there are two points lying on the vertical line which passes through the element $2 \in X$
- R_2 is a function because each vertical line has only one point lying on it , the range of the function R_2 is $\{6, 8, 9\}$
- R_3 is not a function because there is no point on the vertical line which passes through the element $1 \in X$

Example 7

If $X = \{0, 1, 2, 3\}$, $Y = \{2, 3, 4, 5, 6\}$ and R is a relation from X to Y where "a R b" means "a + b = 5" for each $a \in X$, $b \in Y$

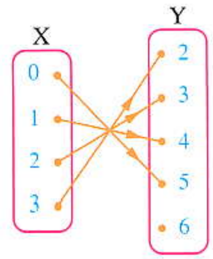
, write the relation R and represent it by an arrow diagram.

Mention giving reasons if R is a function from X to Y or not.

And if it is a function , find its range.

Solution

- $R = \{(0, 5), (1, 4), (2, 3), (3, 2)\}$



- R represents a function from X to Y because each element of X is connected with only one element of Y

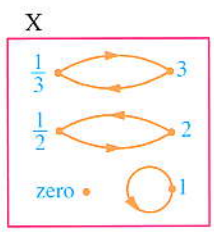
The range of the function = $\{5, 4, 3, 2\}$

Example 8

If $X = \{3, 2, 1, \text{zero}, \frac{1}{2}, \frac{1}{3}\}$ and R is a relation on X where "a R b" means "a is the multiplicative inverse of b" for each $a \in X$, $b \in X$, write R and represent it by an arrow diagram and mention giving reasons if R represents a function or not.

Solution

- $R = \{(3, \frac{1}{3}), (2, \frac{1}{2}), (1, 1), (\frac{1}{2}, 2), (\frac{1}{3}, 3)\}$



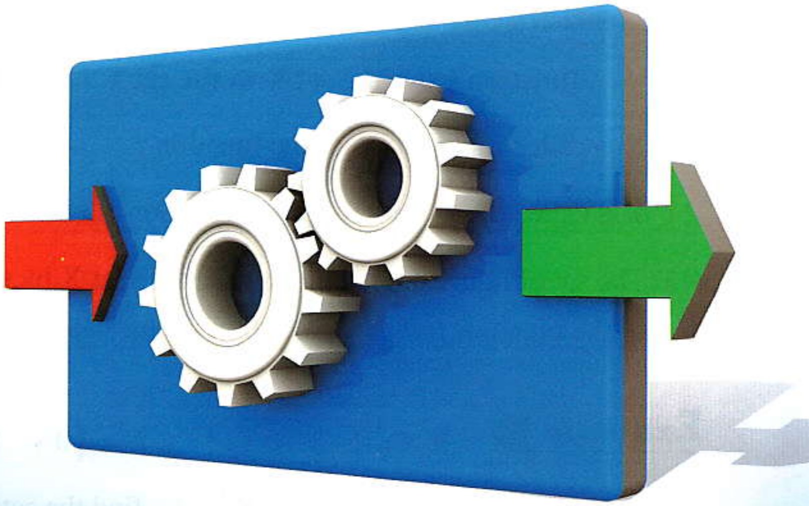
- R does not represent a function because the element zero $\in X$ is not connected with any element in X (There is no arrow going out from zero in the arrow diagram which represents the relation)

TRY by yourself 3

If $X = \{1, 2, 3\}$, $Y = \{1, 4, 6, 9\}$ and R is a relation from X to Y where "a R b" means "$a = \sqrt{b}$" for each $a \in X$, $b \in Y$, write the relation R and represent it by an arrow diagram. Mention giving reasons if R is a function from X to Y or not, and if it is a function, mention its range.

Lesson 3

The symbolic representation of the function - Polynomial functions



The symbolic representation of the function

WATCH VIDEO

- The function is usually denoted by one of the letters f or g or k or ...
  and the function f from the set X to the set Y is written mathematically as :

  $f : X \longrightarrow Y$ and is read as f is a function from X to Y

  or $g : X \longrightarrow Y$ and is read as g is a function from X to Y and so on ...

- If the ordered pair (x , y) belongs to the function , then the element y is called the image of the element x by the function f and we express it by one of the following two forms :

  $f : x \longmapsto y$ and it is read as f maps x to y

  or $f : f(x) = y$ and it is read as f is a function where $f(x) = y$

For example:

If $f : X \longrightarrow Y$ where $f : x \longmapsto x^2$, then $f : 3 \longmapsto 9$

, also can be written in the form : $f(x) = x^2$, hence $f(3) = 9$

! Remark

The mathematical form $f(x) = x^2$ is called the rule of the function f, and it is used to find the image of any element of the domain by the function f

Remember that

If f is a function from the set X to the set Y i.e. $f : X \longrightarrow Y$, then :

1. X is called the **domain** of the function f
2. Y is called the **codomain** of the function f
3. The set of images of the elements of the set X by the function f is called the **range** of the function f which is a subset of the codomain Y

Example 1

If $X = \{-1 , 0 , 1\}$, $Y = \{0 , -1 , -2\}$ and the function $f : X \longrightarrow Y$ where $f(x) = x^2 - 1$, find the set of the function f and represent it by an arrow diagram , then write its range.

Solution

$\because f(x) = x^2 - 1$

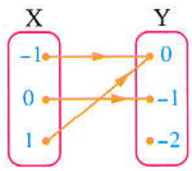
$\therefore f(-1) = (-1)^2 - 1 = 0$ $\qquad \therefore (-1 , 0) \in$ the set of the function f

$, f(0) = (0)^2 - 1 = -1$ $\qquad \therefore (0 , -1) \in$ the set of the function f

$, f(1) = (1)^2 - 1 = 0$ $\qquad \therefore (1 , 0) \in$ the set of the function f

$\therefore$ The set of the function $f = \{(-1 , 0) , (0 , -1) , (1 , 0)\}$

The range of the function $f = \{0 , -1\}$



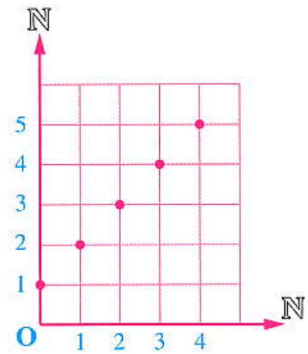
Remark

If f is a function from the set X to itself : i.e. $f : X \longrightarrow X$, then we say «f is a function on X»

Example 2 If $f : \mathbb{N} \longrightarrow \mathbb{N}$ where $\mathbb{N}$ is the set of natural numbers and $f(X) = X + 1$ find $f(0)$, $f(1)$, $f(2)$, $f(3)$ and $f(4)$, then graph a part of the square net of the Cartesian product $\mathbb{N} \times \mathbb{N}$ and represent on it five elements of this function. What is the range of the function f ?

Solution $f(X) = X + 1$ for each $X \in \mathbb{N}$

means that the image of any natural number by the function f is "the number + 1"



$\therefore f(0) = 0 + 1 = 1$

, $f(1) = 2$

, $f(2) = 3$

, $f(3) = 4$

, $f(4) = 5$

$\therefore (0 , 1) , (1 , 2) , (2 , 3) , (3 , 4) , (4 , 5)$

are five elements of f

- The range of f is all the natural numbers except zero. (because there is no natural number added 1 gives zero)

  i.e. The range of $f = \mathbb{N} - \{0\}$

TRY by yourself 1

If $X = \{2 , 4 , 6 , 8\}$

, $Y = \{1 , 2 , 3 , 4 , 5 , 6\}$

and the function $f : X \longrightarrow Y$ where $f(X) = \frac{1}{2} X$

, write the set of the function f and represent it by a Cartesian diagram , then find its range.

Polynomial functions

Definition

The function $f : \mathbb{R} \longrightarrow \mathbb{R}$, $f(X) = a_0 + a_1 X + a_2 X^2 + \cdots + a_n X^n$

where a_0 , a_1 , a_2 , $\cdots$, $a_n \in \mathbb{R}$, $n \in \mathbb{N}$ is called a polynomial function.

i.e. **The polynomial function is a function whose rule is a term or an algebraic expression in condition that the following should be identified :**

1. Each of the domain and the codomain of the function is the set of real numbers.
2. The power (the index) of the variable X in any of its terms is a natural number.

For example: The following functions are all polynomial functions :

- $f : f(X) = 2X + 5$
- $k : k(X) = 8$
- $g : g(X) = X^2 - 2X + 1$
- $n : n(X) = 1 + \sqrt{2}X - 9X^3$

Remark

If the domain or the codomain of a function is not the set of real numbers , then that function is not a polynomial function.

For example :

- $f : f(X) = \sqrt{X}$ is not a polynomial function because $f(X)$ doesn't exist in $\mathbb{R}$ if X equals a negative number.

  For example : $f(-1) \notin \mathbb{R}$ because $\sqrt{-1} \notin \mathbb{R}$

  , so the domain of the function f is not the set of real numbers.

- $h : h(X) = \frac{1}{X}$ is not a polynomial function

  because $h(X)$ doesn't exist in $\mathbb{R}$ if X equals zero. i.e. $h(0) \notin \mathbb{R}$

  , so the domain of the function h is not the set of real numbers.

Remark

When we search if the function is a polynomial or not , we do not simplify its rule.

For example:

The function $f_1 : f_1(X) = X\left(X + \frac{1}{X}\right)$ doesn't represent a polynomial function

because $f_1(0) \notin \mathbb{R}$ while the function $f_2 : f_2(X) = X^2 + 1$ represents a polynomial function.

And notice that: $X\left(X + \frac{1}{X}\right) = X^2 + 1$ for all real numbers except 0

TRY by yourself 2 ***Which of the functions defined by the following rules represents a polynomial function :***

1. $f_1(X) = X(X^2 - 3)$
2. $f_2(X) = X\left(\frac{2}{X} + 5\right)$
3. $f_3(X) = X^2 - \sqrt{X} + 1$
4. $f_4(X) = X^2 - (X^2 - 4)$

The degree of the polynomial function

The degree of the polynomial function is the highest power of the variable in the function rule.

For example:

- The function $f_1 : f_1(X) = 3X - \frac{1}{2}$ is of the first degree (a linear function)
- The function $f_2 : f_2(X) = \sqrt{5}\,X^2 - 3X + 4$ is of the second degree (a quadratic function)
- The function $f_3 : f_3(X) = X^3 - 5X^2 + 4$ is of the third degree (a cubic function)

Remarks

- The function $f : f(X) = a$ where $a \in \mathbb{R} - \{0\}$

  is a polynomial function of zero degree (a constant function) as $f : f(X) = 3$

  In the case of $a = 0$ i.e. When $f(X) = 0$, then the function f has no degree.
- When you want to determine the degree of the function you should simplify its rule to the simplest form before telling its degree.

Example 3 **Choose the correct answer from the given ones :**

1 The function $f : f(x) = x^2 (2 + x)^2$ is a polynomial function of the .... degree.

(a) first (b) second (c) third (d) fourth

2 The function $f : f(x) = x^2 - (x - 5)^2$ is a polynomial function of the .... degree.

(a) zero (b) first (c) second (d) fourth

3 The function $f : f(x) = x^4 - (x^2 + 1)(x^2 - 1)$ is a polynomial function of the .... degree.

(a) zero (b) first (c) second (d) fourth

4 If $f(x) = x^2 - x - 2$, then $f(-3) =$ ....

(a) -3 (b) 4 (c) 10 (d) 14

5 If $f(x) = x^2 - 2x + 5$, then $f(0) =$ ....

(a) 2 (b) 4 (c) 5 (d) 7

6 If $f(x) = x^2 - \sqrt{3}\, x$, then $f\left(-\sqrt{3}\right) =$ ....

(a) 0 (b) 3 (c) 6 (d) $2\sqrt{3}$

7 If $f(x) = x^3$, then $f(3) + f(-3) =$ ....

(a) 54 (b) 27 (c) 6 (d) 0

8 If $f(x) = a x - 6$, $f(2) = 0$, then a = ...

(a) -6 (b) -3 (c) 3 (d) 0

Solution

1 **(d) The reason :** $\because f(x) = x^2 (4 + 4x + x^2) = 4x^2 + 4x^3 + x^4$

$\therefore f$ is a function of the fourth degree.

2 **(b) The reason :** $\because f(x) = x^2 - (x^2 - 10x + 25) = x^2 - x^2 + 10x - 25$

$= 10x - 25$

$\therefore f$ is a function of the first degree.

3 **(a) The reason :** $\because f(x) = x^4 - (x^4 - 1) = x^4 - x^4 + 1 = 1$

$\therefore f$ is a function of the zero degree.

4 **(c) The reason :** Substituting by $x = -3$ at the function rule

$\therefore f(-3) = (-3)^2 - (-3) - 2 = 9 + 3 - 2 = 10$

5 **(c) The reason :** Substituting by $x = 0$ at the function rule

$$\therefore f(0) = 0^2 - 2(0) + 5 = 0 - 0 + 5 = 5$$

6 **(c) The reason :** Substituting by $x = -\sqrt{3}$ at the function rule

$$\therefore f\left(-\sqrt{3}\right) = \left(-\sqrt{3}\right)^2 - \left(\sqrt{3}\right)\left(-\sqrt{3}\right) = 3 + 3 = 6$$

7 **(d) The reason :** $\because f(3) = 3^3 = 27 \quad , \quad f(-3) = (-3)^3 = -27$

$$\therefore f(3) + f(-3) = 27 + (-27) = 0$$

8 **(c) The reason :** $\because f(2) = 0 \qquad \therefore a \times 2 - 6 = 0$

$$\therefore 2a = 6 \qquad \therefore a = 3$$

TRY by yourself 3

Choose the correct answer from the given ones :

1. The function $f : f(x) = x(x^3 - 2)$ is a polynomial function of the .... degree.
   (a) first (b) second (c) third (d) fourth
2. If $f(x) = 3 - 5x$, then $f(-2) =$ ....
   (a) 1 (b) 5 (c) 7 (d) 13
3. If $f(x) = x^2 + x - 1$, then $f(1) + f(-1) =$ ....
   (a) -2 (b) 0 (c) 2 (d) 3
4. If $f(x) = 4x + k$, $f(2) = 15$, then $k =$ ....
   (a) 2 (b) 4 (c) 7 (d) 15

Example 4 If $f(x) = x^2 - 2x + 5$

, **prove that :** $f\left(2\sqrt{2} + 1\right) = 2 f\left(1 - \sqrt{2}\right)$

Solution

$\because f\left(2\sqrt{2} + 1\right) = \left(2\sqrt{2} + 1\right)^2 - 2\left(2\sqrt{2} + 1\right) + 5$

$= 8 + 1 + 4\sqrt{2} - 4\sqrt{2} - 2 + 5 = 12$ (1)

$, f\left(1 - \sqrt{2}\right) = \left(1 - \sqrt{2}\right)^2 - 2\left(1 - \sqrt{2}\right) + 5$

$= 1 + 2 - 2\sqrt{2} - 2 + 2\sqrt{2} + 5 = 6$ (2)

From (1) and (2) : $\therefore f\left(2\sqrt{2} + 1\right) = 2 f\left(1 - \sqrt{2}\right)$

Example 5 If $f(X) = 2X + b$ and $g(X) = X^2 + b$ and if $f(2) + g(-4) = 30$,

then find : $f(-2) - g(2)$

Solution

$\because f(2) = 2 \times 2 + b = 4 + b$, $g(-4) = (-4)^2 + b = 16 + b$

, $\because f(2) + g(-4) = 30$ $\therefore 4 + b + 16 + b = 30$

$\therefore 20 + 2b = 30$ $\therefore 2b = 30 - 20 = 10$

$\therefore b = \frac{10}{2} = 5$

$\therefore f(X) = 2X + 5$, $g(X) = X^2 + 5$

$\therefore f(-2) = 2 \times (-2) + 5 = 1$, $g(2) = 2^2 + 5 = 9$

$\therefore f(-2) - g(2) = 1 - 9 = -8$

TRY by yourself 4 If $f(X) = 2X + 5$ and $g(X) = X - 6$, ***then prove that :*** $f(2) + 3g(3) = 0$

Lesson **4**

The study of some polynomial functions

First The linear function

Definition

The function $f : \mathbb{R} \longrightarrow \mathbb{R}$ where $f(x) = ax + b$, $a \in \mathbb{R} - \{0\}$, $b \in \mathbb{R}$ is called a linear function (it is a polynomial function of the first degree).

Examples of linear functions :

- $f : \mathbb{R} \longrightarrow \mathbb{R}$, $f(x) = x - 1$
- $f : \mathbb{R} \longrightarrow \mathbb{R}$, $f(x) = 2x + 1$
- $f : \mathbb{R} \longrightarrow \mathbb{R}$, $f(x) = 3x$

Notice that :

In each of the shown functions , the index of x is 1 , therefore each of them is a function of the first degree.

The graphical representation of the linear function

- The linear function $f : \mathbb{R} \longrightarrow \mathbb{R}$ where $f(x) = ax + b$, $a \in \mathbb{R} - \{0\}$, $b \in \mathbb{R}$ is represented graphically by a straight line intersecting :

  – The y-axis at the point (0 , b)

  – The x-axis at the point $\left(\frac{-b}{a}, 0\right)$

- To represent a linear function , it is enough to find two ordered pairs belonging to the function.
- You can find a third ordered pair to check that the three points are on the same straight line.

Example 1 **Graph each of the following linear functions :**

1 $f : f(x) = 2x - 3$

2 $r : r(x) = -\frac{1}{2}x$

Solution

1 Determine three ordered pairs belonging to the function.

$\because f(x) = 2x - 3$

$\therefore f(-1) = 2(-1) - 3 = -5$ $\therefore (-1, -5) \in f$

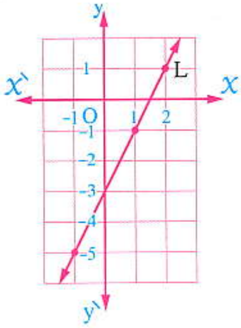
$, f(1) = 2 \times 1 - 3 = -1$ $\therefore (1, -1) \in f$

and $f(2) = 2 \times 2 - 3 = 1$ $\therefore (2, 1) \in f$

You can arrange these ordered pairs in the opposite table :

x	-1	1	2
$y = f(x)$	-5	-1	1

Locate these three points which represents the three ordered pairs in the Cartesian plane and draw the straight line L which passes through any two points of them.
Then check that the third point lies on the same straight line.
Then this straight line is the graphical representation of this function.



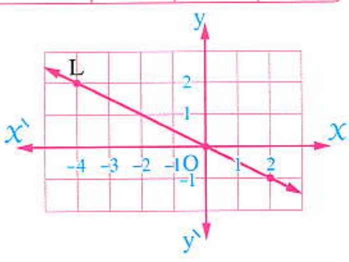
Notice that :

- The point of intersection with y-axis = $(0, b) = (0, -3)$
- The point of intersection with x-axis = $\left(-\frac{b}{a}, 0\right) = \left(\frac{3}{2}, 0\right)$

2 $\because r(x) = -\frac{1}{2}x$ $\therefore$

x	0	2	-4
$y = r(x)$	0	-1	2

From the opposite graph notice that , the straight line L passes through the origin point O (0 , 0)



Generally

The function $f : \mathbb{R} \longrightarrow \mathbb{R}$ where $f(x) = ax$, $a \in \mathbb{R}^*$
is represented graphically by a straight line passing through the origin point (0 , 0)

TRY by yourself 1

Represent graphically each of the following linear functions :

1 $f : f(x) = 3x - 3$

2 $f : f(x) = 2x$

Example 2

1. If the point (a , – a) lies on the straight line representing the function $f : f(\mathcal{X}) = \mathcal{X} - 6$, find the value of a
2. If the straight line representing the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ where $f(\mathcal{X}) = a\mathcal{X} + b$ intersects the y-axis at (0 , 3) and $f(2) = 7$, find the value of each of a , b

Solution

1. ∵ (a , – a) lies on the straight line representing the function f

∴ (a , – a) satisfies the function

∴ $a - 6 = -a$ ∴ $2a = 6$ ∴ $\boxed{a = 3}$

2. ∵ The straight line intersects the y-axis at (0 , 3)

∴ (0 , 3) satisfies the function ∴ $3 = a \times 0 + b$

∴ $\boxed{b = 3}$, ∵ $f(2) = 7$ ∴ $7 = 2a + 3$

∴ $2a = 4$ ∴ $\boxed{a = 2}$

TRY by yourself 2

If the straight line representing the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ where $f(\mathcal{X}) = 4\mathcal{X} - a$ intersects the $\mathcal{X}$-axis at (2 , b) , find the value of each of a , b

Second The constant function

Definition

The function $f : \mathbb{R} \longrightarrow \mathbb{R}$ where $f(\mathcal{X}) = b$, $b \in \mathbb{R}$ is called a constant function.

For example:

$f : f(\mathcal{X}) = 5$ is a constant function where $f(1) = 5$, $f(0) = 5$, $f(-2) = 5$, ... and so on.

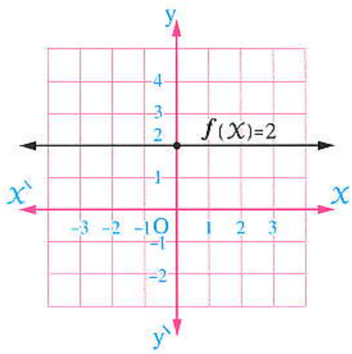
The graphical representation of the constant function

The constant function $f : f(\mathcal{X}) = b$ (where $b \in \mathbb{R}$) is represented by a straight line parallel to $\mathcal{X}$-axis and passing through the point (0 , b) **and this line is :**

- **above** $\mathcal{X}$-axis if $b > 0$
- **below** $\mathcal{X}$-axis if $b < 0$
- **coincident** with $\mathcal{X}$-axis if $b = 0$

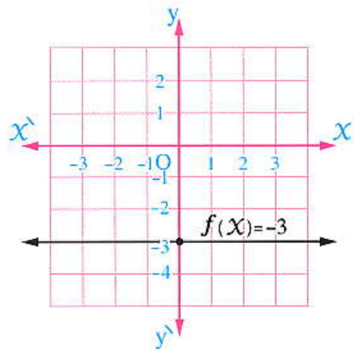
The following examples illustrate that :

$f : f(X) = 2$



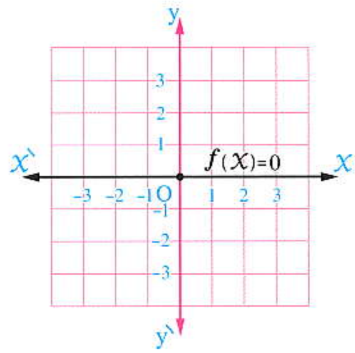
The straight line is above X-axis and passes through (0 , 2)

$f : f(X) = -3$



The straight line is below X-axis and passes through (0 , – 3)

$f : f(X) = 0$



The straight line is coincident with X-axis and passes through (0 , 0)

Example 3

Choose the correct answer from the given ones :

1 The function $f : \mathbb{R} \longrightarrow \mathbb{R}$ where $f(X) = -3$ is represented by a straight line intersecting y-axis at the point ....

(a) (– 3 , 0) (b) (0 , – 3) (c) (3 , 0) (d) (0 , 3)

2 If $f(X) = 4$, then $f(2)$ .... $f(3)$

(a) < (b) > (c) = (d) ≠

3 If $f(X) = 5$, then $2 f(3) =$ ....

(a) 6 (b) $f(6)$ (c) 10 (d) $3 f(2)$

4 If $f(X) = 7$, then $f(7) + f(-7) =$ ....

(a) – 14 (b) – 7 (c) 7 (d) 14

5 If $f(X) = 2$, then $f(X - 2) =$ ....

(a) – 2 (b) 0 (c) 2 (d) 4

Solution

1 **(b)**

2 **(c) The reason :** $\because f$ is a constant function $\therefore f(2) = f(3) = 4$

3 (c) **The reason :** $\because f$ is a constant function $\therefore 2 f(3) = 2 \times 5 = 10$

4 (d) **The reason :** $\because f$ is a constant function

$\therefore f(7) + f(-7) = 7 + 7 = 14$

5 (c) **The reason :** $\because f$ is a constant function $\therefore f(X-2) = f(X) = 2$

TRY by yourself 3

Represent graphically $f : f(X) = -1$, then find the following :

1 The degree of the function f

2 $f(5)$

3 $f(2) + f(-2)$

4 $f(-X)$

Third The quadratic function

Definition

The function $f : \mathbb{R} \longrightarrow \mathbb{R}$ where $f(X) = a X^2 + b X + c$

where a , b and c are real numbers , $a \neq 0$

is called a quadratic function (it is a polynomial function of the second degree).

Examples of quadratic functions :

- $f : \mathbb{R} \longrightarrow \mathbb{R}$, $f(X) = X^2$
- $f : \mathbb{R} \longrightarrow \mathbb{R}$, $f(X) = X^2 - 2$
- $f : \mathbb{R} \longrightarrow \mathbb{R}$, $f(X) = 3 X^2 - 7 X + 2$
- $f : \mathbb{R} \longrightarrow \mathbb{R}$, $f(X) = 6 - X^2 + X$

Notice that :

In each of the shown functions , the highest index of X is 2 , therefore each of them is a function of the 2^{nd} degree.

The graphical representation of the quadratic function

We know that the domain of the quadratic function is the set of real numbers $\mathbb{R}$ which is an infinite set. So , to represent this function graphically , we should represent it on a certain interval by determining some of ordered pairs which belong to the function. Then we draw the curve (paved curve) passing through the points which represent these ordered pairs.

The following examples illustrate that :

Example 4 **Graph each of the following quadratic functions :**

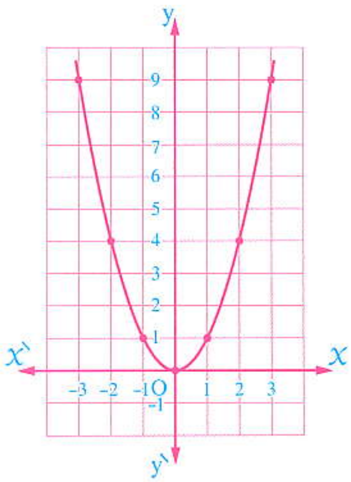
1 $f : f(x) = x^2$, taking $x \in [-3 , 3]$

2 $f : f(x) = -x^2$, taking $x \in [-3 , 3]$

Solution

1 $f(x) = x^2$

x	-3	-2	-1	0	1	2	3
$f(x)$	9	4	1	0	1	4	9



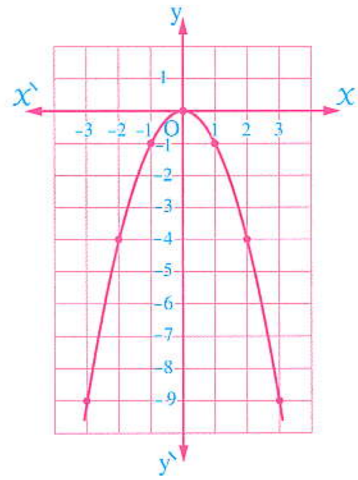
Notice that :

The coefficient of $x^2 > 0$

- The point (0 , 0) is the point of the vertex of the curve , it is considered as **a minimum value** point of the curve because the whole curve **lies up on it**.
- **The minimum** value of the function is zero which is the y-coordinate of the vertex of the curve.
- The curve is symmetric about y-axis
  i.e. The y-axis is the line of symmetry of the curve and its equation is $x = 0$

2 $f(x) = -x^2$

x	-3	-2	-1	0	1	2	3
$f(x)$	-9	-4	-1	0	-1	-4	-9



Notice that :

The coefficient of $x^2 < 0$

- The point (0 , 0) is the point of the vertex of the curve , it is considered as **a maximum value** point of the curve because the whole curve **lies below it**.
- **The maximum** value of the function is zero which is the y-coordinate of the vertex of the curve.
- The curve is symmetric about y-axis
  i.e. The y-axis is the line of symmetry of the curve and its equation is $x = 0$

Generally

The quadratic function $f : f(X) = a X^2 + b X + c$ where a , b and c are real numbers , $a \neq 0$ has the following properties :

1. The vertex of the curve $= \left(\frac{-b}{2a} , f\left(\frac{-b}{2a}\right)\right)$
2. If a (the coefficient of X^2) is positive , then the curve is open upwards and the function has a minimum value equals $f\left(\frac{-b}{2a}\right)$
3. If a (the coefficient of X^2) is negative , then the curve is open downwards and the function has a maximum value equals $f\left(\frac{-b}{2a}\right)$
4. The curve of the function is symmetric about the vertical line which passes through the vertex of the curve and the equation of that line is : $X = \frac{-b}{2a}$ and it is called the axis of symmetry of the curve.

Example 5

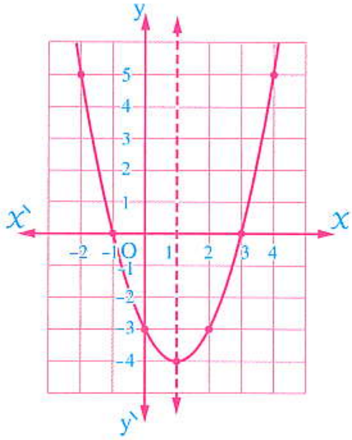
Graph the function $f : f(X) = X^2 - 2X - 3$, taking $X \in [-2 , 4]$, then from the graph , find :

1. The point of the vertex of the curve.
2. The equation of the line of symmetry.
3. The maximum or minimum value of the function.

Solution

$f(X) = X^2 - 2X - 3$

X	−2	−1	0	1	2	3	4
$f(X)$	5	0	−3	−4	−3	0	5



From the graph , we deduce that :

1. The vertex of the curve is (1 , −4)
2. The equation of the line of symmetry is $X = 1$, it is a straight line parallel to y-axis and passing through the vertex of the curve.
3. The minimum value of the function = −4

! Remark

We can form the table used in graphing the function $f : f(x) = x^2 - 2x - 3$ where $x \in [-2, 4]$ by using the scientific calculator which supports (Table) as follows :

1. Turn the calculator on (Table) as follows : Press MODE SETUP , then choose TABLE
2. Input data : Write the rule of the previous function, press successively the following buttons : Start → ALPHA) x^2 − 2 ALPHA) − 3
3. Press the button = , then at the beginning of the interval START? write (−) 2 , then press =
4. At the end of the interval END? write the number 4 , then press =
5. To determine the length of the interval STEP? write 1 , then press =

	x	F(x)
1	−2	5
2	−1	0
3	0	−3
4	1	−4
5	2	−3
6	3	0
7	4	5

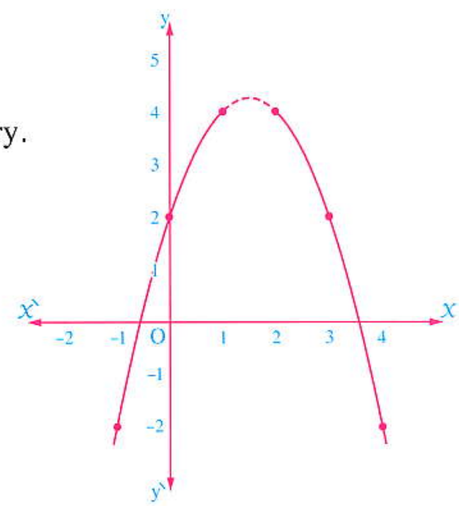
The table is formed in the display, you can move by using button REPLAY up or down.

- To exit the program, press successively the buttons : Start → MODE SETUP 1

Example 6

Graph the function $f : f(x) = -x^2 + 3x + 2$, taking $x \in [-1, 4]$, then find :

1. The maximum value or minimum value of the function.
2. The equation of the line of symmetry.



Solution

x	−1	0	1	2	3	4
$f(x)$	−2	2	4	4	2	−2

When we represent these ordered pairs, we notice that the point of the vertex of the curve is not among these points which makes the drawing of the dotted part in the opposite figure is inaccurate, so the studying of the curve will be difficult, then we should find the vertex point of the curve algebraically as the following :

Finding the vertex point

At the point of the vertex of the curve of the quadratic function , it will be :

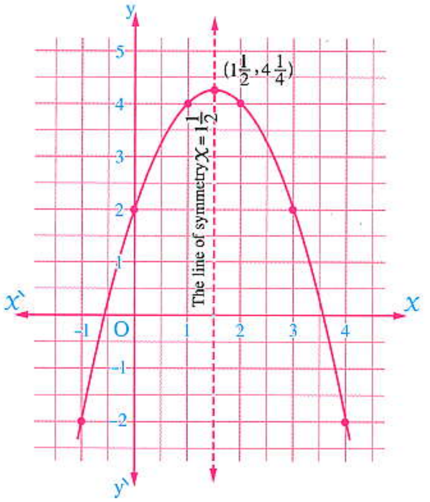
- The X-coordinate $= \frac{-b}{2a}$
- The y-coordinate $= f\left(\frac{-b}{2a}\right)$

where b is the coefficient of X , a is the coefficient of X^2

$\therefore$ X at the vertex of the curve $= \frac{-3}{2 \times -1} = \frac{-3}{-2} = 1\frac{1}{2}$

, $\because f\left(1\frac{1}{2}\right) = -\frac{9}{4} + \frac{9}{2} + 2 = 4\frac{1}{4}$

$\therefore$ The vertex of the curve is $\left(1\frac{1}{2}, 4\frac{1}{4}\right)$



From the vertex of the curve ,

we find that :

1 The maximum value $= 4\frac{1}{4}$

2 The equation of the line of symmetry is $X = 1\frac{1}{2}$

TRY by yourself 4

Graph the curve of the function $f : f(X) = X^2 + 2X - 3$ on the interval $[-4, 2]$

From the graph, find :

1. The maximum or minimum value of the function.
2. The equation of the line of symmetry.

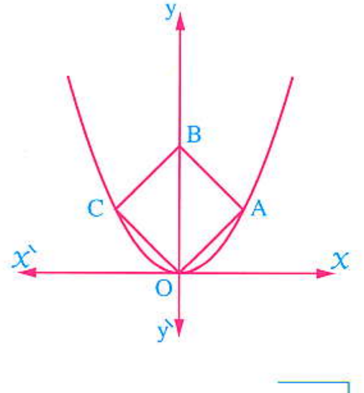
Example 7

In the opposite figure :

ABCO is a square and the curve represents the function $f : f(X) = X^2$

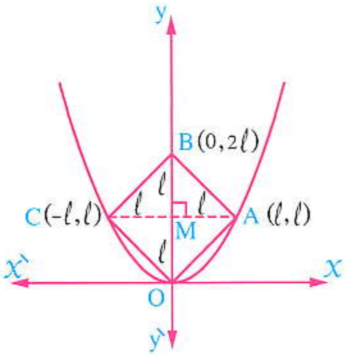
Find the coordinates of the points :

A , B and C



Solution

Draw the square diagonal $\overline{AC}$ to intersect the another diagonal $\overline{BO}$ at the point M



$\because$ The two diagonals of the square are equal in length and bisect each other.

$\therefore$ MA = MB = MC = MO and let : MA = ℓ

$\therefore$ MA = MB = MC = MO = ℓ

$\therefore$ A (ℓ , ℓ) , C $(-\ell , \ell)$, B $(0 , 2\ell)$

$\because$ A $(\ell , \ell) \in$ the function $f : f(x) = x^2$

By substituting in the rule of the function

$\therefore \ell = \ell^2$ $\quad\therefore \ell^2 - \ell = 0$ $\quad\therefore \ell(\ell - 1) = 0$

$\therefore \ell = 0$ (refused) $\quad$ or $\ell - 1 = 0$ $\quad\therefore \ell = 1$

$\therefore$ A (1 , 1) , B (0 , 2) and C (− 1 , 1)

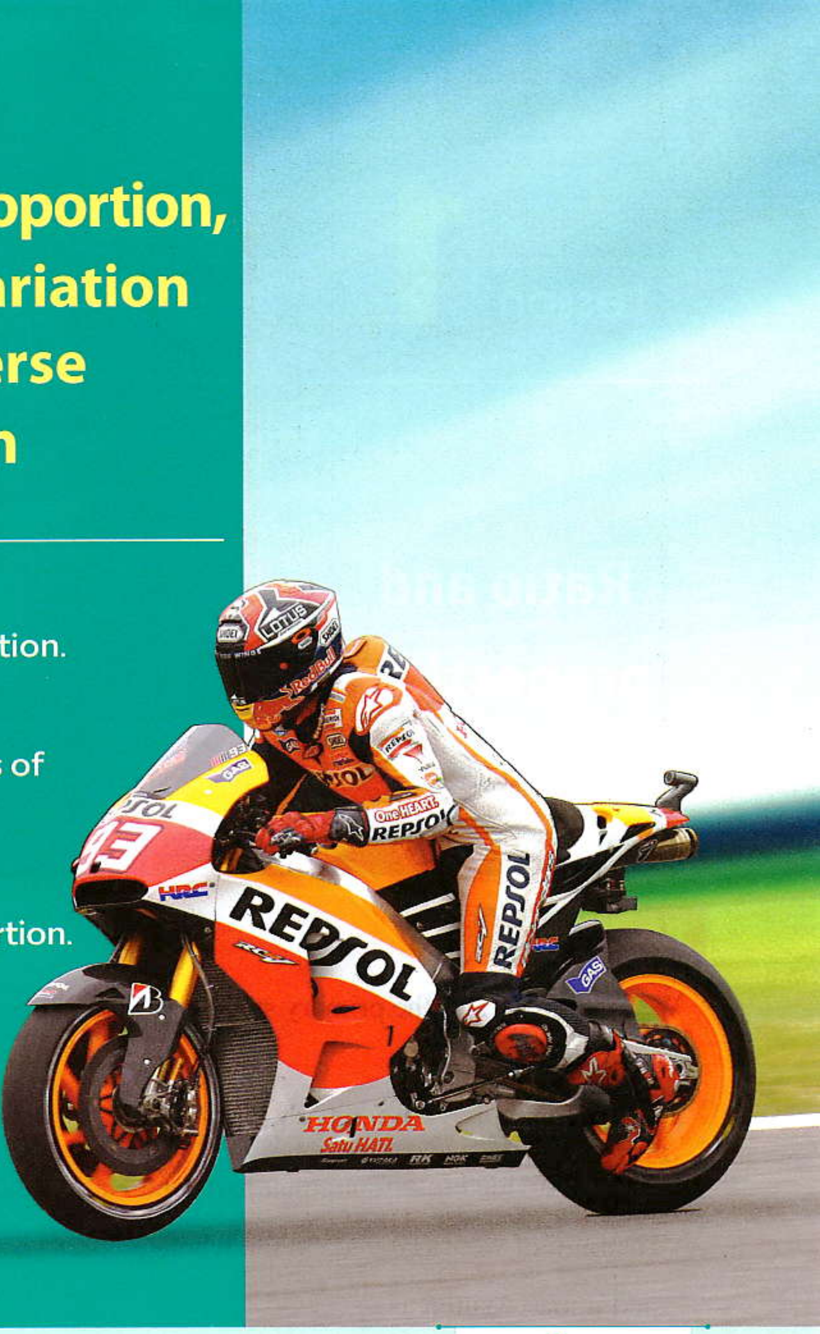
UNIT 2 Ratio, proportion, direct variation and inverse variation

Lesson One Ratio and proportion.

Lesson Two Follow properties of proportion.

Lesson Three Continued proportion.

Lesson Four Direct variation and inverse variation.



Unit Objectives : By the end of this unit, student should be able to :

- recognize the concept of the ratio.
- recognize the properties of the ratio.
- recognize the concept of the proportion.
- recognize the properties of the proportion.
- recognize the concept of the continued proportion.
- use the properties of the ratio and the proportion for solving a lot of problems.
- recognize the concept of the direct variation.
- recognize the concept of the inverse variation.
- differentiate between the direct variation and the inverse variation.
- solve real life problems on the direct variation and the inverse variation.
- appreciate the role of mathematics in solving a lot of real life problems.

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Lesson 1

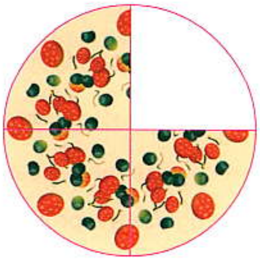
Ratio and proportion

First Ratio

We have studied in the primary stage that the ratio is one of methods of comparison between two quantities.

For example:

If a pie is divided into four equal parts and Hany ate one part only of it , then :



- The ratio of what Hany ate to the whole pie is 1 : 4

  and it may written as $\frac{1}{4}$
- The ratio of what was left of the pie to the whole pie is 3 : 4

  and it may written as $\frac{3}{4}$
- The ratio of what Hany ate to which was left of the pie is 1 : 3

  and it may written as $\frac{1}{3}$

Generally

If a and b are two real numbers , then :
The ratio between a and b is written as a : b or $\frac{a}{b}$
and is read as a to b where :
a is called the antecedent of the ratio , b is called the consequent of the ratio , a and b are called together the two terms of the ratio.

Properties of the ratio

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Property 1

The value of the ratio does not change if each of its terms is multiplied or divided by the same non-zero real number.

i.e.

$a : b = ak : bk \text{ , } k \in \mathbb{R}^*$

For example:

$1 : 2 = 1 \times ④ : 2 \times ④$

i.e. $1 : 2 = 4 : 8$

$a : b = \frac{a}{n} : \frac{b}{n} \text{ , } n \in \mathbb{R}^*$

For example:

$4 : 6 = \frac{4}{②} : \frac{6}{②}$

i.e. $4 : 6 = 2 : 3$

Property 2

The value of the ratio ($\neq 1$) changes if we add or subtract (to or from) each of its two terms a non-zero real number.

i.e.

$a : b \neq a + k : b + k \text{ , } k \in \mathbb{R}^*$ where $a \neq b$

For example:

$3 : 4 \neq 3 + ① : 4 + ①$

i.e. $3 : 4 \neq 4 : 5$

$a : b \neq a - k : b - k \text{ , } k \in \mathbb{R}^*$ where $a \neq b$

For example:

$5 : 8 \neq 5 - ③ : 8 - ③$

i.e. $5 : 8 \neq 2 : 5$

Second Proportion

WATCH VIDEO

The opposite table shows two sets of numbers.

The set (A)	2	4	7	3	6
The set (B)	8	16	28	12	24

If we look at these sets , we can notice that :

$\frac{2}{8} = \frac{4}{16} = \frac{7}{28} = \frac{3}{12} = \frac{6}{24}$ each of them equals $\frac{1}{4}$

In this case , we say that the numbers of set (A) are proportional to the corresponding numbers in the set (B)

The previous form which expresses the equality of two ratios or more is called proportion.

Definition of proportion

It is the equality of two ratios or more.

i.e.

If $\frac{a}{b} = \frac{c}{d}$, then the quantities a , b , c and d are proportional.

And vice versa : If a , b , c and d are proportional , then : $\frac{a}{b} = \frac{c}{d}$

- a is called the **first** proportional.
- b is called the **second** proportional.
- c is called the **third** proportional.
- d is called the **fourth** proportional.

a and **d** are called **extremes** and **b** and **c** are called **means**.

For example: The numbers 1 , 4 , 7 and 28 are proportional numbers , because $\frac{1}{4} = \frac{7}{28}$

And : 1 is the first proportional , 4 is the second proportional , 7 is the third proportional , 28 is the fourth proportional , 1 and 28 are the extremes of this proportion and 4 and 7 are the means.

Properties of proportion

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Property 1

If $\frac{a}{b} = \frac{c}{d}$, then : $a \times d = b \times c$ (the product of the extremes = the product of the means)

The reason : If we multiply each ratio by b d , we get : $\frac{a}{b} \times b\,d = \frac{c}{d} \times b\,d$

i.e. $a \times d = b \times c$

Example 1

Choose the correct answer from the given ones :

1 The third proportional for the quantities 2 , 4 and 20 is ....

(a) 10 (b) 15 (c) 20 (d) 40

2 The fourth proportional for the numbers 4 , 12 and 16 is ....

(a) 24 (b) ± 24 (c) 48 (d) ± 48

3 If 2 , X , 4 and 6 are proportional , then X = ...

(a) 1 (b) 3 (c) 5 (d) 8

Solution

1 **(a) The reason :** Let the third proportional be X

$\therefore$ The quantities 2 , 4 , X and 20 are proportional

$\therefore \frac{2}{4} = \frac{X}{20}$ $\therefore 2 \times 20 = 4 \times X$

$\therefore 40 = 4\,X$ $\therefore X = 10$

2 **(c) The reason :** Let the fourth proportional be X

$\therefore$ The numbers 4 , 12 , 16 and X are proportional

$\therefore \frac{4}{12} = \frac{16}{X}$ $\therefore 4X = 12 \times 16$ $\therefore X = \frac{12 \times 16}{4} = 48$

3 **(b) The reason :** $\because$ 2 , X , 4 and 6 are proportional

$\therefore \frac{2}{X} = \frac{4}{6}$ $\therefore 4X = 12$ $\therefore X = 3$

TRY by yourself 1

If the quantities X , 23 , 15 and 69 are proportional , ***find the value of :*** X

Example 2

Find the number that will be added to each of the numbers : 1 , 13 , 7 and 31 to get proportional numbers.

Solution

Let the number be X $\therefore 1 + X , 13 + X , 7 + X , 31 + X$ are proportional.

$\therefore \frac{1 + X}{13 + X} = \frac{7 + X}{31 + X}$ $\therefore (X + 1)(X + 31) = (X + 7)(X + 13)$

$\therefore X^2 + 32X + 31 = X^2 + 20X + 91$ $\therefore 32X - 20X = 91 - 31$

$\therefore 12X = 60$ $\therefore X = 5$ $\therefore$ The required number = 5

Example 3

If $(2X + 5) : (3X - 3) = 5 : 4$, **find the value of :** X

Solution

$\because \frac{2X + 5}{3X - 3} = \frac{5}{4}$ $\therefore 4(2X + 5) = 5(3X - 3)$

$\therefore 8X + 20 = 15X - 15$ $\therefore 20 + 15 = 15X - 8X$

$\therefore 35 = 7X$ $\therefore X = \frac{35}{7} = 5$

Example 4

Find the number that if we add to the two terms of the ratio 17 : 22 , the result will be 6 : 7

Solution

Let the required number be X $\therefore \frac{17 + X}{22 + X} = \frac{6}{7}$

$\therefore 7(17 + X) = 6(22 + X)$ $\therefore 119 + 7X = 132 + 6X$

$\therefore 7X - 6X = 132 - 119$ $\therefore X$ (The required number) = 13

TRY by yourself 2

Find the real number that if we subtract from both terms of the ratio $\frac{5}{6}$, it will become $\frac{3}{2}$

Property 2

If $a \times d = b \times c$, then $\frac{a}{b} = \frac{c}{d}$

The reason : If we divide each side by b d , we get : $\frac{a \times d}{b\,d} = \frac{b \times c}{b\,d}$ i.e. $\frac{a}{b} = \frac{c}{d}$

Also we can deduce that :

- If $a \times d = b \times c$, then $\frac{a}{c} = \frac{b}{d}$
- If $a \times d = b \times c$, then $\frac{b}{a} = \frac{d}{c}$
- If $a \times d = b \times c$, then $\frac{c}{a} = \frac{d}{b}$

Example 5

In each of the following , find $\frac{x}{y}$ if :

1 $12x = 3y$

2 $4x - 3y = 0$

Solution

1 $\because 12x = 3y$ $\therefore \frac{x}{y} = \frac{3}{12} = \frac{1}{4}$

2 $\because 4x - 3y = 0$ $\therefore 4x = 3y$ $\therefore \frac{x}{y} = \frac{3}{4}$

Example 6

If $4x - 3y : 2x + y = \frac{4}{7}$, find in the simplest form the ratio $x : y$

Solution

$\because \frac{4x - 3y}{2x + y} = \frac{4}{7}$ $\therefore 7(4x - 3y) = 4(2x + y)$

$\therefore 28x - 21y = 8x + 4y$ $\therefore 28x - 8x = 21y + 4y$

$\therefore 20x = 25y$ $\therefore \frac{x}{y} = \frac{25}{20}$ $\therefore \frac{x}{y} = \frac{5}{4}$

Example 7

If $2x^2 - 6y^2 = xy$, **find :** $x : y$

Solution

$\because 2x^2 - 6y^2 = xy$ $\therefore 2x^2 - xy - 6y^2 = 0$

$\therefore (2x + 3y)(x - 2y) = 0$ $\therefore 2x + 3y = 0$

, then $2x = -3y$ $\therefore \boxed{\frac{x}{y} = -\frac{3}{2}}$

or $x - 2y = 0$, then $x = 2y$ $\therefore \boxed{\frac{x}{y} = \frac{2}{1}}$

i.e. $\frac{x}{y} = -\frac{3}{2}$ or $\frac{x}{y} = \frac{2}{1}$

TRY by yourself 3

1. If $2a - 5b = 0$, ***find :*** $\frac{a}{b}$
2. If $\frac{x + 2y}{4x - 3y} = \frac{7}{6}$, ***then prove that :*** $\frac{x}{y} = \frac{3}{2}$
3. If $4a^2 - 9b^2 = 0$, ***find :*** $a : b$

Property 3

If $\frac{a}{b} = \frac{c}{d}$, then $\frac{a}{c} = \frac{b}{d}$

i.e. $\frac{\text{The antecedent of the first ratio}}{\text{The antecedent of the second ratio}} = \frac{\text{The consequent of the first ratio}}{\text{The consequent of the second ratio}}$

The reason : If we multiply each ratio by $\frac{b}{c}$, we get : $\frac{a}{\cancel{b}} \times \frac{\cancel{b}}{c} = \frac{\cancel{c}}{d} \times \frac{b}{\cancel{c}}$

i.e. $\frac{a}{c} = \frac{b}{d}$

For example: If $\frac{a}{4} = \frac{b}{3}$, then $\frac{a}{b} = \frac{4}{3}$ and $\frac{b}{a} = \frac{3}{4}$

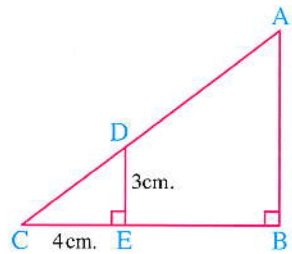
Example 8

In the opposite figure :

ABC is a right-angled triangle at B in which :

$D \in \overline{AC}$, $E \in \overline{BC}$ where $\overline{DE} \perp \overline{BC}$

, DE = 3 cm. and EC = 4 cm. **Find :** AB : BC



Solution In $\Delta\Delta$ ABC , DEC : m ($\angle$ B) = m ($\angle$ DEC) = 90° , $\angle$ C is a common angle

$\therefore$ m ($\angle$ A) = m ($\angle$ EDC)

$\therefore \Delta ABC \sim \Delta DEC$, then we deduce that : $\frac{AB}{DE} = \frac{BC}{EC}$

$\therefore \frac{AB}{3} = \frac{BC}{4}$ $\qquad \therefore \frac{AB}{BC} = \frac{3}{4}$

Property 4

If $\frac{a}{b} = \frac{c}{d}$, then $a = cm$ and $b = dm$ (where m is a constant $\neq 0$)

For example: If $\frac{a}{b} = \frac{3}{4}$, then : $a = 3m$, $b = 4m$ (where m is a constant $\neq 0$)

Example 9 If a : b = 3 : 5 , find the ratio 20 a – 7 b : 15 a + b

Solution $\because \frac{a}{b} = \frac{3}{5}$ $\therefore$ a = 3 m , b = 5 m (where m ≠ 0)

Substituting by a and b in terms of m :

$$\therefore \frac{20a - 7b}{15a + b} = \frac{60m - 35m}{45m + 5m} = \frac{25m}{50m} = \frac{1}{2}$$

Another solution :

By dividing the terms of the ratio $\frac{20a - 7b}{15a + b}$ by b

, then substituting by the value $\frac{a}{b} = \frac{3}{5}$

$$\therefore \frac{20a - 7b}{15a + b} = \frac{20\left(\frac{a}{b}\right) - 7}{15\left(\frac{a}{b}\right) + 1} = \frac{20 \times \frac{3}{5} - 7}{15 \times \frac{3}{5} + 1} = \frac{12 - 7}{9 + 1} = \frac{5}{10} = \frac{1}{2}$$

Example 10 If $\frac{a}{b} = \frac{2}{3}$ and $\frac{x}{y} = \frac{3}{5}$, **prove that :**

(7 a x + 4 b y) , (11 a y + b x) , 12 and 14 are proportional quantities.

Solution $\because \frac{a}{b} = \frac{2}{3}$ $\therefore$ a = 2 m , b = 3 m (where m ≠ 0)

$\because \frac{x}{y} = \frac{3}{5}$ $\therefore$ x = 3 k , y = 5 k (where k ≠ 0)

[**Notice that :** We used two different constants m and k]

Substituting by a , b , x and y

$$\therefore \frac{7ax + 4by}{11ay + bx} = \frac{7 \times 2m \times 3k + 4 \times 3m \times 5k}{11 \times 2m \times 5k + 3m \times 3k}$$

$$= \frac{42mk + 60mk}{110mk + 9mk} = \frac{102mk}{119mk} = \frac{6}{7}$$

$\because \frac{12}{14} = \frac{6}{7}$

$\therefore$ (7 a x + 4 b y) , (11 a y + b x) , 12 and 14 are proportional quantities.

TRY by yourself 4

If $\frac{x}{y} = \frac{2}{5}$, ***prove that :*** (2 x + y) , (x + 2 y) , 12 and 16 are proportional quantities.

Example 11 The ratio between two real numbers is 4 : 7

If we subtract 16 from each of them , then the ratio between the two obtained numbers is 2 : 5 Find the two numbers.

Solution Let the two numbers be a and b

$\therefore \frac{a}{b} = \frac{4}{7}$ $\therefore a = 4\text{ m} , b = 7\text{ m}$ (where $m \neq 0$)

$\therefore \frac{4\text{ m} - 16}{7\text{ m} - 16} = \frac{2}{5}$ $\therefore 14\text{ m} - 32 = 20\text{ m} - 80$

$\therefore 80 - 32 = 20\text{ m} - 14\text{ m}$ $\therefore 48 = 6\text{ m}$ $\therefore m = \frac{48}{6} = 8$

$\therefore a = 4 \times 8 = 32 , b = 7 \times 8 = 56$ **i.e.** The two numbers are 32 and 56

TRY by yourself 5 The ratio between two integers is 2 : 5 If 2 is subtracted from the first integer and 1 is added to the second , then the ratio becomes 1 : 4

Find the two integers.

Lesson 2

Follow properties of proportion

In this lesson , we will study the property (5) from properties of proportion , before studying this property , we will study an important remark in proportion to help us solving problems.

! Important remark

* If a , b , c and d are proportional quantities and we assume that : $\frac{a}{b} = \frac{c}{d} = m$, then

(a) = bm , (c) = dm

For example:

If $\frac{a}{b} = \frac{c}{d} = \frac{3}{4}$, then $a = \frac{3}{4}\, b$, $c = \frac{3}{4}\, d$

* **Generally**

If a , b , c , d , e , f , ... are proportional quantities and we assume that :

$\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \cdots = m$, then (a) = bm , (c) = dm , (e) = fm , ⋯

Example 1 If a , b , c and d are proportional quantities , **prove that :**

1 $\frac{2\,a + 3\,c}{7\,a - 5\,c} = \frac{2\,b + 3\,d}{7\,b - 5\,d}$

2 $\frac{a + c}{b + d} = \frac{a^2 + c^2}{ab + cd}$

Solution

1 Let $\frac{a}{b} = \frac{c}{d} = m$ ∴ (a) = bm , (c) = dm

$$\text{L.H.S.} = \frac{2\,bm + 3\,dm}{7\,bm - 5\,dm} = \frac{m\,(2\,b + 3\,d)}{m\,(7\,b - 5\,d)} = \frac{2\,b + 3\,d}{7\,b - 5\,d} = \text{R.H.S.}$$

2 Let $\frac{a}{b} = \frac{c}{d} = m$ $\therefore$ ⓐ $= bm$, ⓒ $= dm$

$\because \frac{a+c}{b+d} = \frac{bm+dm}{b+d} = \frac{m(b+d)}{(b+d)} = m$ (1)

, $\because \frac{a^2+c^2}{ab+cd} = \frac{(bm)^2+(dm)^2}{bm \times b + dm \times d} = \frac{b^2 m^2 + d^2 m^2}{b^2 m + d^2 m} = \frac{m^2(b^2+d^2)}{m(b^2+d^2)} = m$ (2)

From (1) and (2) we deduce that : $\frac{a+c}{b+d} = \frac{a^2+c^2}{ab+cd}$

Example 2

If a , b , c , d , e and f are positive proportional quantities ,

prove that : $\sqrt{\frac{a^2+c^2+e^2}{b^2+d^2+f^2}} = \frac{a}{b}$

Solution

Let $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = m$ $\therefore$ ⓐ $= bm$, ⓒ $= dm$, ⓔ $= fm$

$\therefore \sqrt{\frac{a^2+c^2+e^2}{b^2+d^2+f^2}} = \sqrt{\frac{(bm)^2+(dm)^2+(fm)^2}{b^2+d^2+f^2}} = \sqrt{\frac{b^2 m^2 + d^2 m^2 + f^2 m^2}{b^2+d^2+f^2}}$

$= \sqrt{\frac{m^2(b^2+d^2+f^2)}{(b^2+d^2+f^2)}} = \sqrt{m^2} = m$

, $\because \frac{a}{b} = m$ $\therefore \sqrt{\frac{a^2+c^2+e^2}{b^2+d^2+f^2}} = \frac{a}{b}$

TRY by yourself 1

If $\frac{a}{b} = \frac{c}{d}$, ***prove that :*** $\frac{5a-2c}{5b-2d} = \frac{4a+3c}{4b+3d}$

Property 5

We know that : $\frac{9}{15} = \frac{6}{10} = \frac{3}{5}$

- If we add the antecedents and consequents of the 1st and the 2nd ratios, we get the ratio $\frac{9+6}{15+10} = \frac{15}{25} = \frac{3}{5}$ which is one of the given ratios.
- Also if we add the antecedents and consequents of the 2nd and the 3rd ratios, we get the ratio $\frac{6+3}{10+5} = \frac{9}{15}$ = one of the given ratios.
- If we add the antecedents and consequents of the three given ratios , we get the ratio $\frac{9+6+3}{15+10+5} = \frac{18}{30} = \frac{3}{5}$ = one of the given ratios.

- Since the ratio does not change if we multiply its two terms by a non-zero real number , then if we multiply the two terms of the first ratio by any number as 2 and multiply the two terms of the second ratio by any other number as (– 4) , then the previous proportion stays true.

i.e. $\frac{18}{30} = \frac{-24}{-40} = \frac{3}{5}$

- If we add the antecedents and consequents of the first and the second ratios , we get the ratio $\frac{18-24}{30-40} = \frac{-6}{-10} = \frac{3}{5}$ = one of the given ratios.
- If we add the antecedents and consequents of the three ratios, we get the ratio $\frac{18-24+3}{30-40+5} = \frac{-3}{-5} = \frac{3}{5}$ = one of the given ratios.

From the previous points , we can say that :
If we have some equal ratios , then we can obtain many other ratios , each of them equals any of the initial ratios. This will happen by adding the antecedents and consequents of all the ratios or some of them directly or after multiplying the two terms of each ratio by a non-zero real number.

i.e.

If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \cdots$ and m_1 , m_2 , m_3 , ⋯ are non-zero real numbers

, then $\frac{m_1 a + m_2 c + m_3 e + \cdots}{m_1 b + m_2 d + m_3 f + \cdots}$ = one of the given ratios.

Remark : The first problem in example (1) can be solved by using the previous property as follows :

∵ a , b , c and d are proportional quantities.

∴ $\frac{a}{b} = \frac{c}{d}$ multiplying the two terms of the 1^{st} ratio by 2 and the 2^{nd} by 3

Then the sum of antecedents : The sum of consequents = one of the given ratios.

∴ $\frac{2a+3c}{2b+3d}$ = one of the given ratios. (1)

Multiplying the two terms of the 1^{st} ratio by 7 and the 2^{nd} by (– 5) then

the sum of antecedents : the sum of consequents = one of the given ratio

∴ $\frac{7a-5c}{7b-5d}$ = one of the given ratios. (2)

From (1) and (2) : ∴ $\frac{2a+3c}{2b+3d} = \frac{7a-5c}{7b-5d}$ ∴ $\frac{2a+3c}{7a-5c} = \frac{2b+3d}{7b-5d}$

Example 3 If $\frac{a}{4} = \frac{b}{5} = \frac{c}{3}$,

find : $\frac{a - b + c}{a + b - c}$

Solution Multiplying the two terms of the 2^{nd} ratio by (– 1) , then add the antecedents and the consequents of the three ratios :

$\therefore \frac{a - b + c}{4 - 5 + 3} = \frac{a - b + c}{2}$ = one of the given ratios. (1)

Multiplying the two terms of the 3^{rd} ratio by (– 1) , then add the antecedents and the consequents of the three ratios :

$\therefore \frac{a + b - c}{4 + 5 - 3} = \frac{a + b - c}{6}$ = one of the given ratios. (2)

From (1) and (2) : $\therefore \frac{a - b + c}{2} = \frac{a + b - c}{6}$

$\therefore \frac{a - b + c}{a + b - c} = \frac{2}{6} = \frac{1}{3}$

Example 4 If $\frac{a + b}{11} = \frac{b + c}{9} = \frac{c + a}{4}$,

prove that : $\frac{a + b + c}{5 a + 4 b + 3 c} = \frac{6}{25}$

Solution Adding the antecedents and consequents of the three ratios.

$\therefore \frac{\text{The sum of antecedents}}{\text{The sum of consequents}}$ = one of the given ratios.

$\therefore \frac{2 a + 2 b + 2 c}{24}$ = one of the given ratios.

$\therefore \frac{a + b + c}{12}$ = one of the given ratios. (1)

Multiplying the two terms of the 1^{st} ratio by 3 and the 3^{rd} by 2 and adding the antecedents and consequents of the three ratios

$\therefore \frac{\text{The sum of antecedents}}{\text{The sum of consequents}}$ = one of the given ratios.

$\therefore \frac{3 a + 3 b + b + c + 2 c + 2 a}{33 + 9 + 8}$ = one of the given ratios.

$\therefore \frac{5 a + 4 b + 3 c}{50}$ = one of the given ratios. (2)

From (1) and (2) :

$\therefore \frac{a + b + c}{12} = \frac{5 a + 4 b + 3 c}{50}$ $\therefore \frac{a + b + c}{5 a + 4 b + 3 c} = \frac{12}{50} = \frac{6}{25}$

Example 5 If $\frac{a + 4b}{x + 2y} = \frac{4b + 7c}{2y + 5z} = \frac{7c + a}{5z + x}$,

prove that : $\frac{a}{2b} = \frac{x}{y}$

Solution Multiplying the two terms of the 2^{nd} ratio by (-1) , then add the antecedents and the consequents of the three ratios :

$\therefore \frac{a + 4b - 4b - 7c + 7c + a}{x + 2y - 2y - 5z + 5z + x} = \frac{2a}{2x} = \frac{a}{x}$ = one of the given ratios. (1)

Multiplying the two terms of the 3^{rd} ratio by (-1) , then add the antecedents and the consequents of the three ratios :

$\therefore \frac{a + 4b + 4b + 7c - 7c - a}{x + 2y + 2y + 5z - 5z - x} = \frac{8b}{4y} = \frac{2b}{y}$ = one of the given ratios. (2)

From (1) and (2) : $\therefore \frac{a}{x} = \frac{2b}{y}$ $\therefore \frac{a}{2b} = \frac{x}{y}$

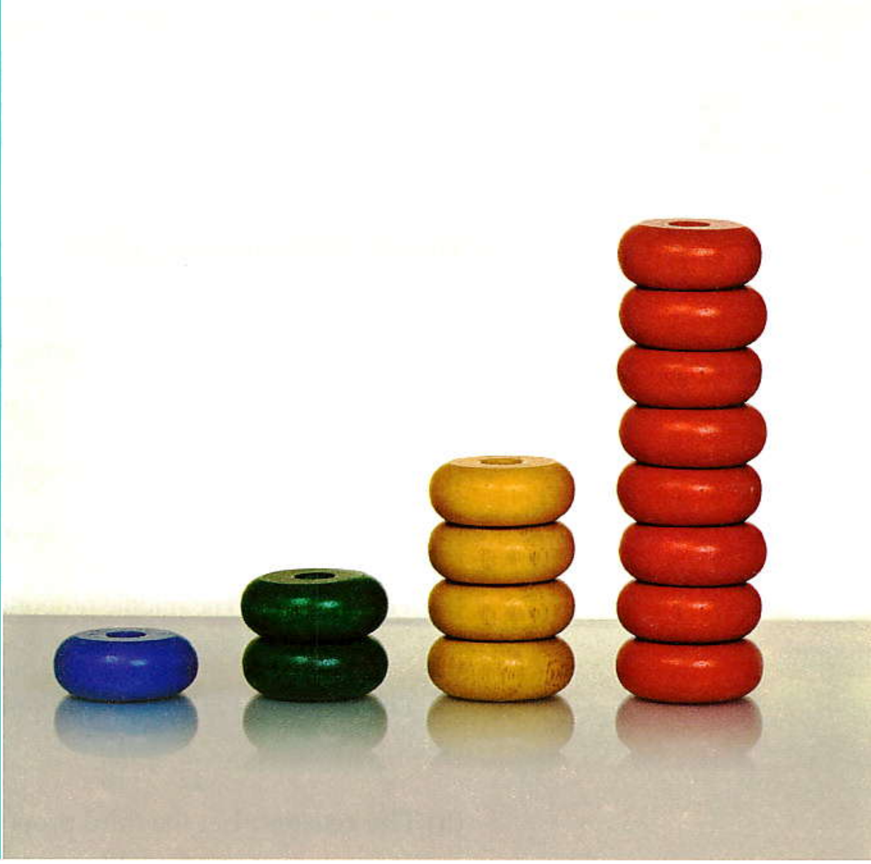
TRY by yourself 2

If $\frac{x}{a - 2b} = \frac{y}{b - 2c} = \frac{z}{c - 2a}$,

prove that : $\frac{x + 2y - z}{3a - 5c} = \frac{y + 2z}{b - 4a}$

Lesson 3

Continued proportion



Definition

The quantities a , b and c are said to be in continued proportion if $\frac{a}{b} = \frac{b}{c}$ or $b^2 = a\,c$

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In this proportion , **a** is called the **first proportional** , **c** is called the **third proportional** and **b** is called the **middle proportional (proportional mean)**.

For example:

The numbers 4 , 6 and 9 form a continued proportion because : $\frac{4}{6} = \frac{6}{9}$ or because : $(6)^2 = 4 \times 9$ where 6 is the middle proportional , 4 is the first proportional and 9 is the third proportional.

Notice that :

1. If a , b and c are in continued proportion , then : $b^2 = a\,c$ i.e. $b = \pm\sqrt{ac}$ and the two quantities a and c should be either both positive or both negative.
2. For any two positive numbers or any two negative numbers x and y , there are two middle proportional ($\sqrt{xy}$ and $-\sqrt{xy}$)

Example 1

Choose the correct answer from the given ones :

1. The middle proportional between 5 and 20 is ....
   (a) – 10   (b) 10   (c) ± 10   (d) 100
2. The middle proportional between 3 and $\frac{1}{3}$ is ....
   (a) ± 1   (b) 9   (c) $\frac{1}{9}$ (d) ± 9
3. The middle proportional between $3x^3$ and $27x$ is ....
   (a) $9x^2$ (b) $\pm 9x^2$ (c) $9x^4$ (d) $\pm 9x^4$

4 The first proportional of 12 and 18 is ....

(a) 8 (b) ± 8 (c) 12 (d) 27

5 The third proportional of – 6 and 12 is ....

(a) – 24 (b) 6 (c) 18 (d) 72

Solution

1 **(c) The reason :** The middle proportional $= \pm\sqrt{5 \times 20} = \pm\sqrt{100} = \pm 10$

2 **(a) The reason :** The middle proportional $= \pm\sqrt{3 \times \frac{1}{3}} = \pm\sqrt{1} = \pm 1$

3 **(b) The reason :** The middle proportional $= \pm\sqrt{3x^3 \times 27x} = \pm\sqrt{81x^4}$ $= \pm 9x^2$

4 **(a) The reason :** Let the first proportional be a

$\therefore \frac{a}{12} = \frac{12}{18}$ $\therefore a = \frac{12 \times 12}{18} = 8$

5 **(a) The reason :** Let the third proportional be c

$\therefore \frac{-6}{12} = \frac{12}{c}$ $\therefore c = \frac{12 \times 12}{-6} = -24$

TRY by yourself 1

1 ***Find the middle proportional between 32 and 18***

2 ***Find the first proportional of 8 and 16***

! Remark

If a , b and c are in continued proportion and we assume that : $\frac{a}{b} = \frac{b}{c} = m$

, then $\frac{b}{c} = m$ $\therefore$ (b) $= cm$ (1)

, $\because \frac{a}{b} = m$ $\therefore a = bm$

Substituting for b from (1) : $\therefore a = (cm)\, m$ $\therefore$ (a) $= cm^2$

i.e.

If $\frac{a}{b} = \frac{b}{c} = m$, then $\begin{cases} b = cm \\ a = cm^2 \end{cases}$

Example 2 If a , b and c are in continued proportion ,

prove that : $\frac{4a^2 - 3b^2}{4b^2 - 3c^2} = \frac{a}{c}$

Solution Let $\frac{a}{b} = \frac{b}{c} = m$ $\therefore$ (b) $= cm$, (a) $= cm^2$

$\therefore \frac{4a^2 - 3b^2}{4b^2 - 3c^2} = \frac{4(cm^2)^2 - 3(cm)^2}{4(cm)^2 - 3c^2} = \frac{4c^2m^4 - 3c^2m^2}{4c^2m^2 - 3c^2} = \frac{c^2m^2(4m^2 - 3)}{c^2(4m^2 - 3)} = m^2$ (1)

, $\because \frac{a}{c} = \frac{cm^2}{c} = m^2$ (2)

From (1) and (2) , we deduce that : $\frac{4a^2 - 3b^2}{4b^2 - 3c^2} = \frac{a}{c}$

Another solution :

$\because \frac{a}{b} = \frac{b}{c}$ $\therefore b^2 = ac$

$\therefore$ L.H.S. $= \frac{4a^2 - 3ac}{4ac - 3c^2} = \frac{a(4a - 3c)}{c(4a - 3c)} = \frac{a}{c} =$ R.H.S.

Example 3 If b is the middle proportional between a and c , **prove that :**

1. $\frac{a - b}{a} = \frac{a - c}{a + b}$
2. $ab - c^2 = (b - c)(a + b + c)$

Solution $\because$ b is the middle proportional between a and c

$\therefore$ a , b and c are in continued proportion

Let $\frac{a}{b} = \frac{b}{c} = m$

$\therefore$ (b) $= cm$, (a) $= cm^2$

1 $\because \frac{a - b}{a} = \frac{cm^2 - cm}{cm^2} = \frac{cm(m - 1)}{cm^2} = \frac{m - 1}{m}$ (1)

, $\frac{a - c}{a + b} = \frac{cm^2 - c}{cm^2 + cm} = \frac{c(m^2 - 1)}{cm(m + 1)} = \frac{c(m - 1)(m + 1)}{cm(m + 1)} = \frac{m - 1}{m}$ (2)

From (1) and (2) , we deduce that :

$\frac{a - b}{a} = \frac{a - c}{a + b}$

2 $\because ab - c^2 = cm^2 \times cm - c^2 = c^2m^3 - c^2 = c^2(m^3 - 1)$ (1)

, $(b - c)(a + b + c) = (cm - c)(cm^2 + cm + c)$

$= c(m - 1) \times c(m^2 + m + 1)$

$= c^2(m - 1)(m^2 + m + 1) = c^2(m^3 - 1)$ (2)

From (1) and (2) , we deduce that : $ab - c^2 = (b - c)(a + b + c)$

TRY by yourself 2 If a , b and c are in continued proportion , ***prove that :*** $\frac{3c^2 - 4b^2}{3b^2 - 4a^2} = \frac{c^2}{b^2}$

Generalizing the definition of the continued proportion

The quantities a , b , c , d , ... are in continued proportion if : $\frac{a}{b} = \frac{b}{c} = \frac{c}{d} = \cdots$

For example:

The numbers 16 , 24 , 36 and 54 are in continued proportion

because : $\frac{16}{24} = \frac{24}{36} = \frac{36}{54}$ $\left(\text{each ratio} = \frac{2}{3}\right)$

! Remark

If a , b , c and d are in continued proportion and we assume that : $\frac{a}{b} = \frac{b}{c} = \frac{c}{d} = m$, then :

$\frac{c}{d} = m$ $\quad \therefore \boxed{c} = dm$ $\quad$ (1)

, $\frac{b}{c} = m$ $\quad \therefore b = cm$

Substituting for c from (1) : $\therefore b = (dm)\, m$ $\quad \therefore \boxed{b} = dm^2$ $\quad$ (2)

, $\frac{a}{b} = m$ $\quad \therefore a = bm$

Substituting for b from (2) : $\therefore a = (dm^2)\, m$ $\quad \therefore \boxed{a} = dm^3$

i.e.

If $\frac{a}{b} = \frac{b}{c} = \frac{c}{d} = m$, then $\boxed{c = dm}$, $\boxed{b = dm^2}$ and $\boxed{a = dm^3}$

Example 4

If a , b , c and d are in continued proportion

, **prove that :** $\frac{a+d}{b-c+d} = \frac{a-c}{b-c}$

Solution

Let $\frac{a}{b} = \frac{b}{c} = \frac{c}{d} = m$ $\quad \therefore \boxed{c} = dm$, $\boxed{b} = dm^2$, $\boxed{a} = dm^3$

$$\therefore \frac{a+d}{b-c+d} = \frac{dm^3+d}{dm^2-dm+d} = \frac{d\,(m^3+1)}{d\,(m^2-m+1)}$$

$$= \frac{(m+1)\,(m^2-m+1)}{m^2-m+1} = m+1 \qquad (1)$$

$$, \frac{a-c}{b-c} = \frac{dm^3-dm}{dm^2-dm} = \frac{dm\,(m^2-1)}{dm\,(m-1)} = \frac{(m-1)\,(m+1)}{(m-1)} = m+1 \qquad (2)$$

From (1) and (2) , we deduce that : $\frac{a+d}{b-c+d} = \frac{a-c}{b-c}$

TRY by yourself 3

If a , b , c and d are in continued proportion , ***prove that :*** $\frac{a+2b}{b+2c} = \frac{c+a}{d+b}$

Example 5 If the quantities a , 2 b , 3 c and 4 d are in continued proportion ,

prove that : (2 b – 3 c) is the middle proportional between (a – 2 b) and (3 c – 4 d)

Solution

Let $\frac{a}{2b} = \frac{2b}{3c} = \frac{3c}{4d} = m$ $\therefore$ $3c = 4\,dm$, $2b = 4\,dm^2$, $a = 4\,dm^3$

Proving that : (2b – 3c) is the middle proportional between

(a – 2b) and (3c – 4d)

means proving that : $(2b - 3c)^2 = (a - 2b)(3c - 4d)$

$\because (2b - 3c)^2 = (4\,dm^2 - 4\,dm)^2$

$= \left(4\,dm\,(m-1)\right)^2 = 16\,d^2\,m^2\,(m-1)^2$ (1)

, $(a - 2b)(3c - 4d) = (4\,dm^3 - 4\,dm^2)(4\,dm - 4d)$

$= 4\,dm^2\,(m-1) \times 4d\,(m-1) = 16\,d^2m^2\,(m-1)^2$ (2)

From (1) and (2) , we deduce that : $(2b - 3c)^2 = (a - 2b)(3c - 4d)$

$\therefore$ (2b – 3c) is the middle proportional between (a – 2b) and (3c – 4d)

Another solution :

$\because$ a , 2b , 3c and 4d are in continued proportion.

$\therefore \frac{a}{2b} = \frac{2b}{3c} = \frac{3c}{4d}$

Subtracting the terms of the 2nd ratio from the terms of the 1st ratio

$\therefore \frac{a - 2b}{2b - 3c}$ = one of the given ratios. (1)

Subtracting the terms of the 3rd ratio from the terms of the 2nd ratio

$\therefore \frac{2b - 3c}{3c - 4d}$ = one of the given ratios. (2)

From (1) and (2) , we deduce that : $\frac{a - 2b}{2b - 3c} = \frac{2b - 3c}{3c - 4d}$

$\therefore$ (2b – 3c) is the middle proportional between (a – 2b) and (3c – 4d)

Lesson 4

Direct variation and inverse variation

First The direct variation

WATCH VIDEO

Definition

It is said that y varies directly as x and it is written $y \propto x$ if $y = m\,x$

i.e. $\frac{y}{x} = m$, where m is a constant $\neq 0$

, the relation : $y = m\,x$ is represented graphically by a straight line passing through the origin point (0 , 0)

For example:

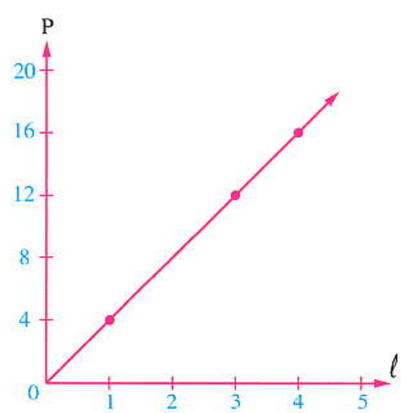
The perimeter of the square (P) is varying directly with its side length (ℓ) and it is written as $P \propto \ell$

Because : $P = 4\,\ell$ or $\frac{P}{\ell} = 4$

and the following table shows some values of ℓ and the values of P corresponding to them.

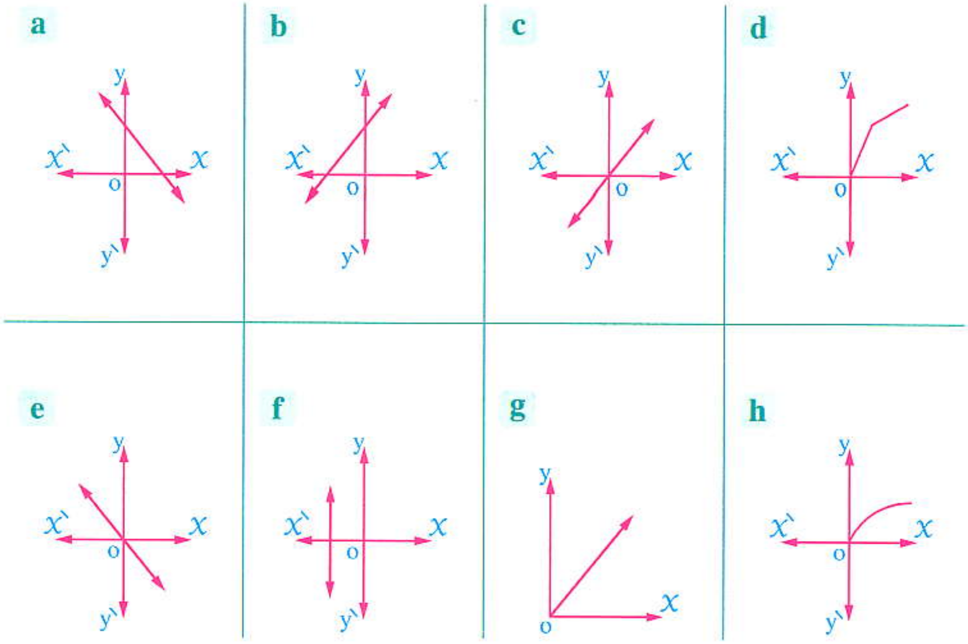
Side length (ℓ)	1	3	4
The perimeter (P)	4	12	16

and the opposite figure represents graphically the relation between P and ℓ



Example 1

Show which of the following graphs represents a direct variation between x and y :



Solution

The graphs which represent a direct variation between x and y are :

c , e and g because in each of them , the straight line passes through the origin point.

Example 2

If $a^2 + 4b^2 = 4ab$, **prove that :** $a \propto b$

Solution

To prove that $a \propto b$ we prove that $a = m\,b$ where m is a constant $\neq 0$

$\because a^2 + 4b^2 = 4ab$ $\quad\therefore a^2 - 4ab + 4b^2 = 0$

$\therefore (a - 2b)^2 = 0$ $\quad\therefore a - 2b = 0$

$\therefore \boxed{a = 2b}$ $\quad\therefore a \propto b$

TRY by yourself 1

If $\frac{3x - 5y}{3x - 9y} = \frac{1}{2}$ for every values of $x \in \mathbb{R}_+$, $y \in \mathbb{R}_+$, ***prove that :*** $x \propto y$

Property

If $y \propto x$, the variable x took the two values x_1 and x_2 and y took the two values y_1 and y_2 respectively , then : $\boxed{\frac{y_1}{y_2} = \frac{x_1}{x_2}}$

The reason : $\because y \propto x$ then $y = m x$ where m is a constant $\neq 0$

at $x = x_1$, $y = y_1$ then $y_1 = m x_1$ (1)

, at $x = x_2$, $y = y_2$ then $y_2 = m x_2$ (2)

Dividing (1) by (2) : $\therefore \frac{y_1}{y_2} = \frac{m x_1}{m x_2}$ $\therefore \frac{y_1}{y_2} = \frac{x_1}{x_2}$

Example 3

If $y \propto x$ and $y = 20$ when $x = 7$

, then find the value of y when $x = 14$

WATCH VIDEO

Solution

$\because y \propto x$ $\therefore \frac{y_1}{y_2} = \frac{x_1}{x_2}$

where $y_1 = 20$, $x_1 = 7$, $y_2 = ?$, $x_2 = 14$

$\therefore \frac{20}{y_2} = \frac{7}{14}$ $\therefore y_2 = \frac{20 \times 14}{7} = 40$

Another solution :

$\because y \propto x$ $\therefore y = m x$ (m is a constant $\neq 0$)

$\because y = 20$ as $x = 7$ $\therefore 20 = m \times 7$

$\therefore m = \frac{20}{7}$ $\therefore y = \frac{20}{7} x$

, when $x = 14$ $\therefore y = \frac{20}{7} \times 14$ $\therefore y = 40$

Example 4

If x and y are two variables where y varies directly as the multiplicative inverse of $\frac{1}{x^3}$, $y = 18$ when $x = 2$

, find the relation between x and y , then find the values of y when $x \in \{0 , 1 , 4\}$

Solution

$\because y \propto$ the multiplicative inverse of $\frac{1}{x^3}$

$\therefore y \propto x^3$ $\therefore y = m x^3$ where m is a constant $\neq 0$

$\because y = 18$ as $x = 2$ $\therefore 18 = m \times (2)^3$ $\therefore m = \frac{18}{8} = \frac{9}{4}$

$\therefore y = \frac{9}{4} x^3$ This is the relation between x and y

as $x = 0$ $\therefore y = \frac{9}{4} \times 0 = 0$

as $x = 1$ $\therefore y = \frac{9}{4} \times 1 = \frac{9}{4} = 2\frac{1}{4}$

as $x = 4$ $\therefore y = \frac{9}{4} \times 64 = 144$

Example 5 If (V) denotes the volume of a right circular cone , its height is constant and if (V) varies directly as the square of radius length of the base of the cone (r) and the volume of the cone was 477 cm.^{3} , when the radius length of its base = 15 cm.

Find the volume of the cone when the base radius length = 10 cm.

Solution $\because V \propto r^2$ $\quad \therefore \frac{V_1}{V_2} = \frac{r_1^{\ 2}}{r_2^{\ 2}}$ $\quad \therefore \frac{V_1}{V_2} = \left(\frac{r_1}{r_2}\right)^2$

where $V_1 = 477$ cm.^{3} , $r_1 = 15$ cm. , $V_2 = ?$, $r_2 = 10$ cm.

$\therefore \frac{477}{V_2} = \left(\frac{15}{10}\right)^2 = \frac{9}{4}$ $\quad \therefore V_2 = \frac{477 \times 4}{9} = 212$ cm.^{3}

TRY by yourself 2 If $x \propto y$ and $y = 2$ when $x = 40$, find the value of x when $y = 3$

Second The inverse variation

WATCH VIDEO

Definition

It is said that y varies inversely as x and it is written $y \propto \frac{1}{x}$ if $y = \frac{m}{x}$

i.e. $x\,y = m$, where m is a constant $\neq 0$

For example:

The uniform velocity (v) varies inversely as time (t) when the covered distance (d) is constant

Because : $v = \frac{d}{t}$ or $vt = d$

, in this case we say that the velocity varies directly as the multiplicative inverse of time and it is written as : $v \propto \frac{1}{t}$

Example 6 If $a^2 b^4 - 10\,ab^2 = -25$, **prove that :** a varies inversely as b^2

Solution To prove that a varies inversely as b^2 we prove that : $ab^2 = m$ where $m \neq 0$

$\because a^2 b^4 - 10\,ab^2 = -25$ $\quad \therefore a^2 b^4 - 10\,ab^2 + 25 = 0$

$\therefore (ab^2 - 5)^2 = 0$ $\quad \therefore ab^2 - 5 = 0$

$\therefore ab^2 = 5$ $\quad \therefore$ a varies inversely as b^2

TRY by yourself 3 If $a^2 b^2 + 49 = 14\,ab$, ***prove that :*** $a \propto \frac{1}{b}$

Property

If $y \propto \frac{1}{x}$, the variable x took the two values x_1 and x_2 and as a result for that y took the two values y_1 and y_2 respectively, then : $\boxed{\frac{y_1}{y_2} = \frac{x_2}{x_1}}$

The reason : $\because y \propto \frac{1}{x}$, then $y = \frac{m}{x}$ where m is a constant $\neq 0$

at $x = x_1$, $y = y_1$, then $y_1 = \frac{m}{x_1}$ (1)

, at $x = x_2$, $y = y_2$, then $y_2 = \frac{m}{x_2}$ (2)

Dividing (1) by (2) :

$$\therefore \frac{y_1}{y_2} = \frac{m}{x_1} \div \frac{m}{x_2} = \frac{\cancel{m}}{x_1} \times \frac{x_2}{\cancel{m}} = \frac{x_2}{x_1}$$

Example 7 If the length of a rectangle (ℓ) varies inversely as its width (w), when the area is constant and $\ell = 12$ cm. as $w = 8$ cm., **find :** ℓ when $w = 3$ cm.

Solution

$\because \ell \propto \frac{1}{w}$

$\therefore \frac{\ell_1}{\ell_2} = \frac{w_2}{w_1}$, where $\ell_1 = 12$ cm., $w_1 = 8$ cm., $\ell_2 = ?$, $w_2 = 3$ cm.

$\therefore \frac{12}{\ell_2} = \frac{3}{8}$ $\therefore \ell_2 = \frac{8 \times 12}{3} = 32$ cm.

Another solution :

$\because \ell \propto \frac{1}{w}$

$\therefore \ell w = m$, where m is a constant $\neq 0$

$\because \ell = 12$ cm. as $w = 8$ cm. $\therefore m = 12 \times 8 = 96$ $\therefore \ell w = 96$

When $w = 3$ cm. $\therefore 3\ell = 96$ $\therefore \ell = \frac{96}{3} = 32$ cm.

Example 8

If y varies inversely as x and $y = 6$ as $x = 2.5$, find the relation between x and y , then find the value of y if $x = 5$

Solution

$\because y \propto \frac{1}{x}$ $\therefore x y = m$, where m is a constant $\neq 0$

$\because y = 6$ as $x = 2.5$ $\therefore m = 6 \times 2.5 = 15$

$\therefore$ The relation between x and y is $\boxed{x y = 15}$

, at $x = 5$ $\therefore 5 y = 15$ $\therefore y = \frac{15}{5} = 3$

Example 9

If $y = 1 + b$ where b varies inversely as x^2 and $y = 17$ as $x = \frac{1}{2}$, find the relation between x and y , then find the value of y when $x = 2$

Solution

$\because b \propto \frac{1}{x^2}$ $\therefore b = \frac{m}{x^2}$, where m is a constant $\neq 0$ $\therefore y = 1 + \frac{m}{x^2}$

, $\because y = 17$ as $x = \frac{1}{2}$ $\therefore 17 = 1 + \frac{m}{\left(\frac{1}{2}\right)^2}$ $\therefore 17 = 1 + \frac{m}{\frac{1}{4}}$

Subtracting 1 from both sides : $\therefore 16 = \frac{m}{\frac{1}{4}}$

$\therefore m = 16 \times \frac{1}{4} = 4$ $\therefore \boxed{y = 1 + \frac{4}{x^2}}$

at $x = 2$: $\therefore y = 1 + \frac{4}{2^2} = 1 + \frac{4}{4} = 2$

TRY by yourself 4

If y varies inversely as x and $y = 2$ as $x = 6$, calculate the value of y as $x = 1$

UNIT 3

Statistics

Lesson One Collecting data.

Lesson Two Dispersion.

Unit Objectives : By the end of this unit, student should be able to :

- recognize the different resources of collecting data.
- recognize the methods of collecting data , and the advantages and the disadvantages of each method.
- recognize the concept of the sample.
- recognize the methods of selection of samples.
- recognize the types of the samples.
- choose the best method to select a sample for studying a certain phenomenon.
- use the calculator and the computer for generating random numbers used in the samples.
- recognize the dispersion measurements.
- recognize the advantages and the disadvantages of the range as one of the dispersion measurements.
- calculate the range of a set of individuals.
- calculate the standard deviation of a set of individuals.
- calculate the standard deviation of a simple frequency distribution.
- calculate the standard deviation of a frequency distribution of sets.
- use the calculator to calculate the standard deviation.

Lesson 1

Collecting data

- The statistical investigator collects , classifies , represents and analyses data in purpose of deducing some results on which he depends in making the suitable decisions.
- The more data is accurate , the more the decisions will be true and reliable.
- Collecting data in such scientific methods will lead to get accurate outcomes when doing operations of statistical inference and proper decision making.
- Collecting statistical data demands knowing the resources of collecting it and determining the methods of collecting it.

Resources of collecting data is classified into

1. **Primary resources (field resources) :**
   These are the resources from which we get data directly.
2. **Secondary resources (historical resources) :**
   These are the resources from which we get data that previously collected and registered by some authorities , formal organisations or persons.

There are some examples for each resource with representing the advantages and the disadvantages of each one :

	1 Primary resources	2 Secondary resources
Examples :	• Personal interview. • Questionnaires (survey). • Observing and measuring.	• Central agency for public mobilization and statistics. • Mass-media and internet. • Documents of data of employees in a company.
Advantages :	Accuracy.	Saves time , effort and money.
Disadvantages :	It needs more time, effort and money besides it requires more investigators in large societies.	It is less accurate.

Methods of collecting data

- The method of collecting data depends on the aim of collecting these data and it also depends on the size of the statistical society under study.
- The statistical society is defined as all individuals which have general common characters.

For example:

- The workers in a factory represent a statistical society , whose individual is the worker.
- The pupils of a school represent a statistical society , whose individual is the pupil.

We will show two methods of collecting data :

1 Method of mass population :

It is based on collecting the data related to the phenomenon under study from all individuals of the statistical society.

2 Method of samples :

It is based on collecting data related to the phenomenon under study from a representative sample of the society , and applying the research on it , then generalizing the results on the whole society.

There are some examples for each method with representing the advantages and the disadvantages of each one :

	1 Method of mass population	2 Method of samples
Examples :	• Elections. • Census. • Setting up a data base of all employees in an organization.	• A sample of a patient's blood to make some clinical check up. • A sample of some products of a factory to find out if it matches the standard specifications.
Advantages :	• Accuracy. • Inclusiveness. • Neutrality. • Representing all the society individuals.	• Saving time , effort and money. • It is the only method for collecting data about large unlimited societies such as the search on contents of the desert sand. • It is the only method for collecting data about some limited societies in which mass population method leads to a great loss in it such as checking a sample of a patient's blood because of checking the whole blood of the patient leads to death.
Disadvantages :	• Sometimes it needs long time , great effort and a great cost.	• The results sometimes are not accurate specially if the sample doesn't represent the statistical society authentically , in this case the sample is called a biased sample.

In the following , we will explain the concept of the sample and its types and how we select it :

The concept of the sample

It is a small part from a large society that looks like the society and represents it well.

How can we select the sample ?

The types of samples according to the method of selecting it

- **The biased selection**
  not a random sample or deliberate sample
- **The randomly selection**
  random sample
  - **Simple** random sample (in case of the homogeneous societies)
  - **Layer** random sample (in case of the heterogeneous societies)

At the following , we explain each type in details :

First The biased selection (samples are not randomly selected)

- It means that we select the sample in a way to satisfy the objectives of the research. This is called **the deliberate sample**.

For example:

If we want to know how the students understood a lesson in algebra , we must analyze the outcomes of the test by considering the outcomes of a group of students studying the same topic without the other students , this is not a random selection.

- The biased selection is not representing the statistical society.

Second Random selection (random samples)

It means to select a sample such that every member of the population has an equal chance of having selected.

The following are the most important types of the random samples which are :

1 Simple random sample. 2 Layer random sample.

1 Simple random sample

- It is used for the homogeneous societies which are not naturally divided into groups or classes.
- It is selected by two ways according to the number of individuals of statistical society as the following.

A The first method : If the size of the society is small :

- **This method will be carried out as follows :**

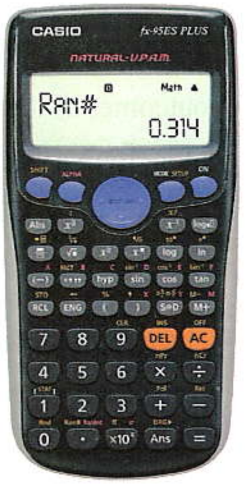
1 Each individual of the society takes a number , this number is written on a card such that all cards are identical.
i.e. There is no difference in colour or size.

2 Each card is folded well such that the number does not appear , then they are put in a box and mixed well.

3 We select the sample by drawing one card from the box blindly , then we turned well the cards and select the next card , and so on till we reach the required number of the sample.

This method is suitable if , for example , we select a sample of 10 workers from a factory that has 50 workers.

B The second method : If the size of the society is large :



In this method , every individual of the society has a number , then we select the sample using the property of the random number in the scientific calculator as in the opposite picture.

- We press the following keys respectively from the left :

then a decimal will appear on the display in the field from 0.000 to 0.999

- If we get a 1-decimal digit , add two zeroes to make it a part of 1000
  For example: (0.2 → 0.200)
- If we get a 2-decimal digit , add one zero to make it a part of 1000

For example: (0.64 → 0.640) and so on.

- Take the number neglecting the decimal point , then the individual who has this number is selected as a member of the sample , then repeat pressing on [=] to get more numbers.
- We will ignore the numbers which are greater than the number of society under study.
- And we ignore the repeated numbers which we selected before.
- The percentage 10% of the number of the society is suitable for holding the survey.

This method is more suitable for selecting a sample of 25 students from a school that has 900 students.

2 Layer random sample

- It is used in the statistical societies which are heterogeneous or made up of qualitative sets that are different in characteristics.
- In this case , we cannot select the sample by the simple random sample method because the sample will not represent the society well because it will not represent all the classes of the society.

Therefore we have to follow the following steps :

1. We divide the society into homogeneous sets according to the characteristics forming it , each set is called **a layer**.
2. We find the number of individuals of each layer , then we find its ratio referring to the total number of the society.
3. To form a sample , we select from each layer a certain number of individuals such that the ratio that represents each layer in the sample is the same ratio of the layer in the whole society , and this by using the following law :

The number of individuals of the layer in the sample

$$= \frac{\text{the total number of individuals in the layer}}{\text{the total number of individuals in the society}} \times \text{the number of individuals of the sample}$$

«approximating the result to the nearest unit»

For example:

When we want to study the educational level of the students of a school of 500 students (boys and girls) and if the ratio between the number of boys to the number of girls is 1 : 4 and we want to select a sample formed from 50 students , we should select 10 students from boys and 40 students from girls , for the sample representing all the society well.

Example 1 A factory has 300 workers. The people in charge of the monthly magazine of this factory want to develop this magazine by doing a survey of a sample representing 10% of the total number of the workers in this factory. Show how the selection of this sample can be carried out using the calculator.

Solution The number of workers in the factory = 300 workers.

$\therefore$ The number of the random sample = $\frac{10}{100} \times 300 = 30$ workers.

Then we want to select 30 workers to hold this survey.

The selection operation can be carried out as follows :

1. Each worker in the factory is given a number from 1 to 300
2. Use the calculator to select 30 numbers randomly , such that these numbers are included between 0 and 301 and the number that is above 300 should be ignored.

For example:

By pressing the keys ON → SHIFT → Ran# (.) → = successively from left to right.

- If we get the decimal 0.049 , then the number of the selected person is 49
- If we get the decimal 0.132 , then the number of the selected person is 132
- If we get the decimal 0.12 , then the number of the selected person is 120
- If we get the decimal 0.453 , it must be ignored because 453 is above 300 and so on till we get 30 numbers.
- Assuming that the calculator gave us the shown numbers in the opposite table , then the workers who carry these numbers are the selected sample to carry out this survey.

49	132	120	141	249	272
254	256	4	213	74	198
131	2	156	47	172	13
8	3	85	82	9	38
41	14	34	279	118	103

Example 2 A factory produced 200 TV sets from the type A , 300 TV sets from the type B and 500 TV sets from the type C , if we want to select a layer sample formed from 50 TV sets such that it represents all the types to examine them.

Calculate the number of TV sets which should be selected from each kind.

Solution

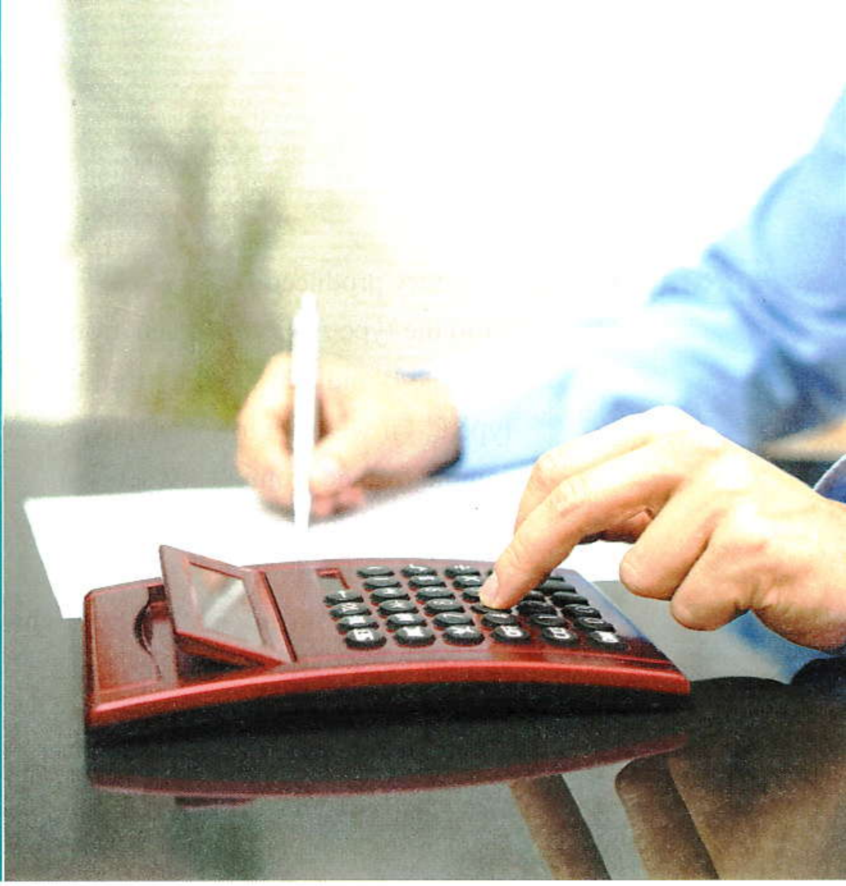
- The total number of TV sets = 200 + 300 + 500 = 1000 TV sets.
- The number of TV sets of the type A in the sample = $\frac{200}{1000} \times 50 = 10$ TV sets.
- The number of TV sets of the type B in the sample = $\frac{300}{1000} \times 50 = 15$ TV sets.
- The number of TV sets of the type C in the sample = $\frac{500}{1000} \times 50 = 25$ TV sets.

TRY *by yourself*

A school has 300 male students and 500 female students wanted to do a survey on a sample of 24 male and female students representing each layer according to its size. Calculate the number of students of each layer in the sample.

Lesson 2

Dispersion



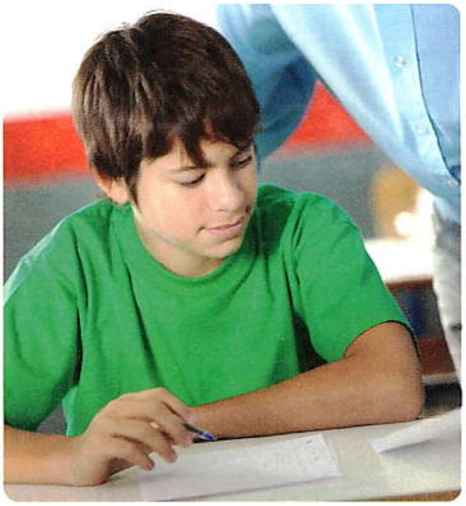
- You studied before some of statistical measures which were known as **"measures of central tendency"** as the mean , the median and the mode.
- And we know that each of them describe the frequency distributions and the statistical data by identifying one numerical value , where the left values centralize about it.
- But in some cases the measures of central tendency are not enough to describe clearly the data.

To explain that , let's study the following case :

Two sets of 5 students each , an exam of maximum mark 50 marks is given for each sets , the marks of the students were as follows :

The set A : 29 , 26 , 35 , 35 , 35

The set B : 8 , 35 , 49 , 35 , 33



At calculating the mean , the median and the mode of the marks of the students in each set alone , we find the shown results in the following table :

	mean	median	mode
Set A	32	35	35
Set B	32	35	35

Remember that

- The mean = $\frac{\text{the sum of values}}{\text{the number of this values}}$
- The median of a set of values is the value which lies at the middle of the set of values after ordering them.
- The mode of a set of values is the most common value in the set.

- In the previous case , the two sets are different , and in spite of that , we found that they have the same mean , median and mode , which don't mean that these sets are necessarily homogeneous.
- Therefore , the measures of central tendency only are unable to describe all the characteristics a set of frequency distributions and statistical data.

  So we need besides the measures of central tendency that depends on determining one value that the other data centralize around it, another kind of measures which depends on determining a degree of convergence or divergence of data.

For example:

In the previous example , the marks of the set A are convergent because their values are included between 26 and 35 marks while the marks of the set B are divergent because their values are included between 8 and 49 marks.

i.e. The marks of the set B are more divergent than the marks of the set A

- These new measures are called the measures of dispersion. We will study each of the range and the standard deviation.

Dispersion of a set of values

It means the divergence or the differences among its values.

- The dispersion is small if the difference among the values is little while the dispersion is great if the difference among the values is great , the dispersion is zero if all the values are equal.

i.e. The dispersion of a set of values is a measure of the degree to which these values spread out and that expresses how much the sets are homogeneous.

Dispersion measurements

1 The range (the simplest measure of dispersion) :

It is the difference between the greatest value and the smallest value in the set.

The range = the greatest value – the smallest value

For example:

- If the values of set A are 60 , 58 , 62 , 61 and 59

  ∴ The range = 62 – 58 = 4

- If the values of set B are 72 , 78 , 46 , 65 and 39

  ∴ The range = 78 – 39 = 39

  So the set B is more divergent than the set A

The advantages of range :

- It is an easy and simple method that gives a quick idea about the divergence or convergence of the values.
- It is considered as the simplest and the easiest method to measure dispersion.

The disadvantages of range :

- It does not reflect the influence of all values because its measure depends on the greatest and smallest values only , therefore it does not give a full idea of the dispersion of the set of values.
- It is influenced greatly by the outlier.

  For example:

  - The range of the set of values : 21 , 22 , 61 , 24 and 26 is (61 – 21 = 40)
  - While if we ignore the value 61 from the set , then the range becomes (26 – 21 = 5)

  i.e. The range equals $\frac{1}{8}$ the previous range , therefore the range is an approximated measure and we cannot depend on it.

2 Standard deviation :

It is the most important , common and accurate measure of dispersion. We can calculate it by calculating the positive square root of the average of squares of deviations of the values from their mean. It is denoted by σ and it is read as (sigma).

First Calculating the standard deviation of a set of values :

The standard deviation $\sigma = \sqrt{\frac{\Sigma (x - \overline{x})^2}{n}}$

Where :

x denotes a value of the values ,

$\overline{x}$ denotes the mean of the values and it is read as x bar ,

n denotes the number of values ,

Σ denotes the summation operation.

Example 1 Calculate the standard deviation of the values : 8 , 9 , 7 , 6 and 5

Solution

1 We find the mean of the values $(\overline{x}) = \frac{\Sigma x}{n} = \frac{8+9+7+6+5}{5} = 7$

2 We form the opposite table :

x	$x - \overline{x}$	$(x - \overline{x})^2$
8	$8 - 7 = 1$	1
9	$9 - 7 = 2$	4
7	$7 - 7 = 0$	0
6	$6 - 7 = -1$	1
5	$5 - 7 = -2$	4
Total		**10**

3 We calculate the standard deviation as follows :

The standard deviation $(\sigma) = \sqrt{\frac{\Sigma (x - \overline{x})^2}{n}} = \sqrt{\frac{10}{5}} = \sqrt{2} \simeq 1.41$

TRY by yourself 1

If 25 , 24 , 25 , 30 , 28 and 30 represent the marks of one of the pupils in examination of algebra in different months , ***find :***

1 The mean.

2 The standard deviation.

Second Calculating the standard deviation of a frequency distribution :

For any frequency distribution : The standard deviation $\sigma = \sqrt{\dfrac{\sum (X - \overline{X})^2 k}{\sum k}}$

Where :

X represents the value or the centre of the set ,

k represents the frequence of the value or the set ,

$\sum k$ is the sum of frequences and $\overline{X}$ (the mean) $= \dfrac{\sum (X \times k)}{\sum k}$

A Calculating the standard deviation of a simple frequency distribution :

Example 2 **The following table shows the distribution of ages of 20 persons in years :**

The age	15	20	22	23	25	30	**Total**
Number of persons	2	3	5	5	1	4	**20**

Find the standard deviation of the ages.

Solution

1 We find the mean of the ages $(\overline{X})$ by using the following table :

The age (X)	**Number of persons (k)**	$X \times k$
15	2	30
20	3	60
22	5	110
23	5	115
25	1	25
30	4	120
Total	**20**	**460**

The mean $(\overline{X}) = \dfrac{\sum (X \times k)}{\sum k} = \dfrac{460}{20} = 23$ years.

2 We form the following table :

x	k	$x-\overline{x}$	$(x-\overline{x})^2$	$(x-\overline{x})^2 \times k$
15	2	$15-23=-8$	64	128
20	3	$20-23=-3$	9	27
22	5	$22-23=-1$	1	5
23	5	$23-23=0$	0	0
25	1	$25-23=2$	4	4
30	4	$30-23=7$	49	196
Total	**20**			**360**

3 We calculate the standard deviation as follows :

Standard deviation $(\sigma) = \sqrt{\dfrac{\Sigma(x-\overline{x})^2 \times k}{\Sigma k}} = \sqrt{\dfrac{360}{20}} = \sqrt{18} \simeq 4.24$ years.

TRY by yourself 2

The following frequency distribution shows the number of days of absentees in a class :

Number of absence days	0	1	2	3	4	Total
Number of pupils	5	7	7	5	6	**30**

Calculate the mean and the standard deviation for the number of days of absence.

B Calculating the standard deviation of a frequency distribution of sets :

Example 3 **The following is the frequency distribution of weekly incentives of 100 workers in a factory :**

Incentives in pounds	35 –	45 –	55 –	65 –	75 –	85 –
Number of workers	10	14	20	28	20	8

Find the standard deviation of this distribution.

Solution

1 We find the mean $(\overline{X})$ by using the following table :

Remember that

The centre of the set $= \frac{\text{lower limit} + \text{upper limit}}{2}$

Sets	Centres of sets (X)	Frequence (k)	$X \times k$
35 –	40	10	400
45 –	50	14	700
55 –	60	20	1200
65 –	70	28	1960
75 –	80	20	1600
85 –	90	8	720
Total		**100**	**6580**

$\therefore$ The mean $(\overline{X}) = \frac{\Sigma (X \times k)}{\Sigma k} = \frac{6580}{100} = 65.8$ pounds.

2 We form the following table :

X	k	$X - \overline{X}$	$(X - \overline{X})^2$	$(X - \overline{X})^2 \times k$
40	10	$40 - 65.8 = -25.8$	665.64	6656.4
50	14	$50 - 65.8 = -15.8$	249.64	3494.96
60	20	$60 - 65.8 = -5.8$	33.64	672.8
70	28	$70 - 65.8 = 4.2$	17.64	493.92
80	20	$80 - 65.8 = 14.2$	201.64	4032.8
90	8	$90 - 65.8 = 24.2$	585.64	4685.12
Total	**100**			**20036**

3 We calculate the standard deviation as follows :

Standard deviation $(\sigma) = \sqrt{\frac{\Sigma (X - \overline{X})^2 \times k}{\Sigma k}} = \sqrt{\frac{20036}{100}} \simeq 14.15$ pounds.

! Remarks

- The standard deviation is influenced by all values not by the two terminal values only (the smallest and the greatest value) as the range , therefore it represents the dispersion better than the range.
- The standard deviation has the same measuring units of the original data.
- The values which are more homogeneous have less dispersion and their standard deviation is small.
- If the standard deviation equals zero that means the all values are equal , it is the perfect homogeneous case (the vanished dispersion)

TRY by yourself 3

For the following frequency distribution , calculate :

1 The mean. 2 The standard deviation.

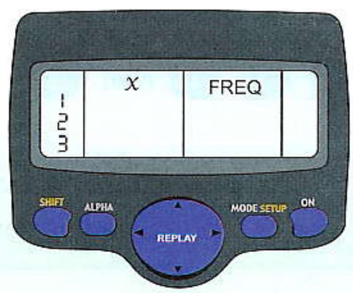
Sets	1 –	3 –	5 –	7 –	9 – 11
Frequency	7	3	5	3	2

Using the calculator to calculate the standard deviation :

- We can use the calculator CASIO ($f\,X$ – 82 ES , $f\,X$ – 85 ES , $f\,X$ – 500 ES, $f\,X$ – 95 ES Plus , $f\,X$ – 991 ES Plus) to calculate the standard deviation.
- The following steps show how to solve the previous example (example 3) using the calculator :
- We will use the calculator ($f\,X$ – 95 ES Plus)

Step (1)

Before inserting the data of the previous example, we should set the calculator system by pressing the following keys from left :



Then the screen will appear as in the opposite figure.

Step (2)

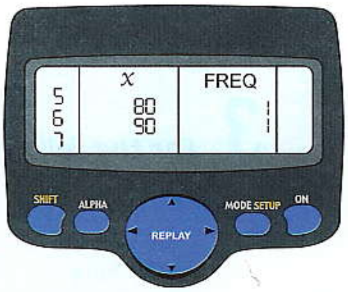
- We insert the values (X) in the case of simple frequency distribution or the centres of sets (X) in the case of frequency distribution of sets in the first column (X)
- With respect to the previous example :

  We insert the centres of sets :

  40 , 50 , 60 , 70 , 80 and 90 by pressing the following keys from left as follows :

  → 4 0 = 5 0 = 6 0 =

  7 0 = 8 0 = 9 0 =



Then the screen will appear as in the opposite figure.

Step (3)

Use the key REPLAY to move to the second column (FREQ), then insert frequencies

10 , 14 , 20 , 28 , 20 and 8 by pressing the following keys from left as follows :

1 0 = 1 4 = 2 0 = 2 8 = 2 0 = 8 = AC

Thus we insert the data of the previous example on the calculator.

Step (4)

For finding the value of the standard deviation , we press the following keys from left :

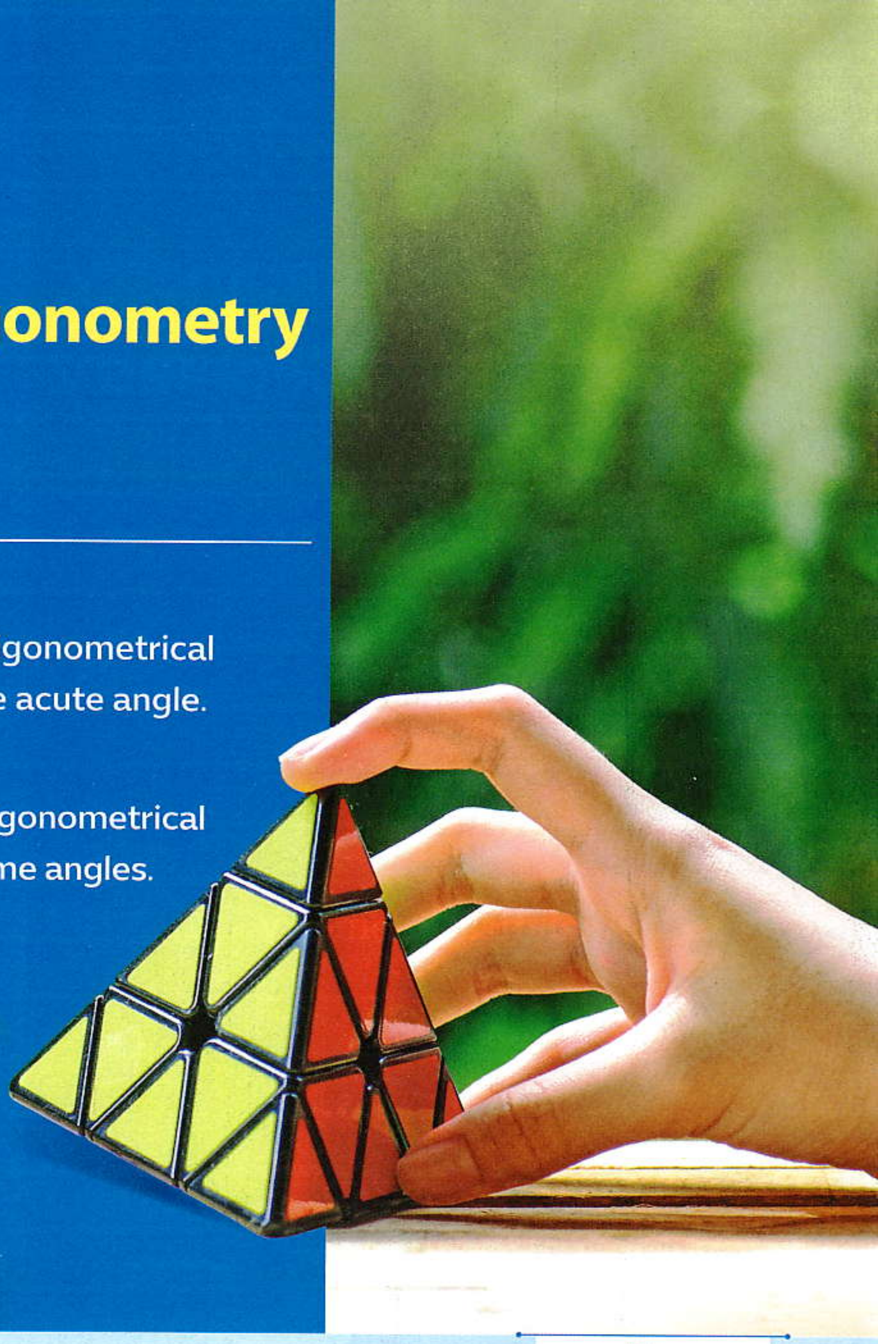
SHIFT 1 4 (VAR) 3 (σx) =

Then the screen will appear as in the opposite figure.

$\therefore$ Standard deviation $\sigma \simeq 14.15$

Second | Trigonometry and Geometry

UNIT 4 Trigonometry



Lesson One The main trigonometrical ratios of the acute angle.

Lesson Two The main trigonometrical ratios of some angles.

Unit Objectives : By the end of this unit, student should be able to :

- recognize the main trigonometrical ratios of the acute angle.
- recognize the main trigonometrical ratios of the angles of measures 30° , 60° and 45°
- find the main trigonometrical ratios of a given angle.
- find the measure of an angle if one of its trigonometrical ratios is given.
- use the calculator to find the main trigonometrical ratios.

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Enriching information :

- Trigonometry is one of mathematics branches and it is one of the general geometry branches , it concerneds studying the relations between the sides and angles of the triangle and the trigonometric ratios as the sine and cosine of the angle.
- Ancient Egyptians were the first to use the trigonometric theorems and rules in building pyramids and temples.
- Trigonometry has many applications in surveying roads and manufactoring motors , TV sets , football playgrounds , calculating geographic distances and astronomy discovering.

Lesson **1**

The main trigonometrical ratios of the acute angle

Prelude

- **You studied before the units of the degree measure of the angle which are :**

The degree which is denoted by 1° , the minute which is denoted by $1'$ and the second which is denoted by $1''$

For example:

The angle whose measure is 22 degrees , 36 minutes and 48 seconds is written as $22^\circ\ 36'\ 48''$

The relation between the degrees, the minutes and the seconds

- $1^\circ = 60'$
- $1' = 60''$

i.e. $1^\circ = 60 \times 60 = 3600''$

Example 1

1 Write in degrees : $22^\circ\ 36'\ 48''$

2 Write in degrees , minutes and seconds : 45.18°

Solution

1 Convert the minutes into degrees , as the following :

$36' = \frac{36}{60} = 0.6^\circ$

Convert the seconds into degrees , as the following :

$48'' = \frac{48}{3600} = 0.01\dot{3}^\circ$

Remember that

$0.00\dot{3}$ is read as the recurring decimal 0.003

i.e. $22^\circ\ 36'\ 48'' = 22^\circ + 0.6^\circ + 0.01\dot{3}^\circ = 22.61\dot{3}^\circ$

Another solution by using the scientific calculator :

Press the keys in sequence from left as follows :

[2] [2] [°'''] [3] [6] [°'''] [4] [8] [°'''] [=] [°''']

Then the result will be 22.61333333

2 Convert 0.18° into minutes as the following : $0.18 \times 60 = 10.8'$

Convert $0.8'$ into seconds as the following : $0.8 \times 60 = 48''$

i.e. $45.18^\circ = 45^\circ\ 10'\ 48''$

Another solution by using the scientific calculator :

Press the keys in sequence from left as follows :

[4] [5] [.] [1] [8] [=] [°''']

Then the result will be $45^\circ\ 10'\ 48''$

Example 2

If the ratio between the measures of two complementary angles is 7 : 9 , find the degree measure of each of them.

Solution

Let the measures of the two angles be :

$7x$ and $9x$

$\therefore 7x + 9x = 90^\circ$

$\therefore 16x = 90^\circ$

$\therefore x = \frac{90^\circ}{16} = 5.625^\circ$

$\therefore$ The measure of the first angle

$= 5.625^\circ \times 7 = 39.375^\circ$

$= 39^\circ\ 22'\ 30''$

, the measure of the second angle $= 5.625^\circ \times 9 = 50.625^\circ = 50^\circ\ 37'\ 30''$

Remember that

- The sum of measures of two complementary angles = 90°
- The sum of measures of two supplementary angles = 180°
- The sum of measures of the interior angles of any triangle = 180°

TRY by yourself 1

If the ratio between the measures of two supplementary angles is 5 : 11 , find the degree measure of each of them.

The main trigonometrical ratios of the acute angle

WATCH VIDEO

The trigonometrical ratio of the acute angle

It is the ratio between two side lengths of the right-angled triangle that contains this angle.

There are three main trigonometrical ratios of the acute angle and they are :

1 **The sine of the angle :**

abbreviated (**sin**) and equals $\dfrac{\text{the length of the } \textbf{opposite} \text{ side to the angle}}{\text{the length of the hypotenuse}}$

2 **The cosine of the angle :**

abbreviated (**cos**) and equals $\dfrac{\text{the length of the } \textbf{adjacent} \text{ side to the angle}}{\text{the length of the hypotenuse}}$

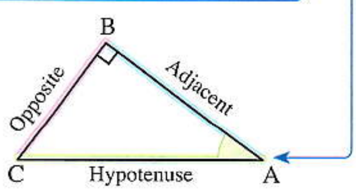
3 **The tangent of the angle :**

abbreviated (**tan**) and equals $\dfrac{\text{the length of the } \textbf{opposite} \text{ side to the angle}}{\text{the length of the } \textbf{adjacent} \text{ side to the angle}}$

i.e.

If Δ ABC is a right-angled triangle at B , then :

According to angle A

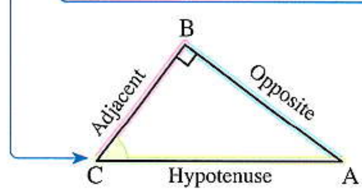


1 $\sin A = \dfrac{\text{Opposite}}{\text{Hypotenuse}} = \dfrac{BC}{AC}$

2 $\cos A = \dfrac{\text{Adjacent}}{\text{Hypotenuse}} = \dfrac{AB}{AC}$

3 $\tan A = \dfrac{\text{Opposite}}{\text{Adjacent}} = \dfrac{BC}{AB}$

According to angle C



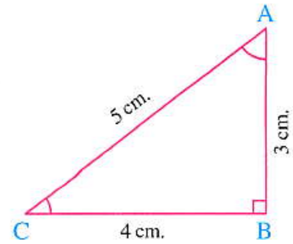
1 $\sin C = \dfrac{\text{Opposite}}{\text{Hypotenuse}} = \dfrac{AB}{AC}$

2 $\cos C = \dfrac{\text{Adjacent}}{\text{Hypotenuse}} = \dfrac{BC}{AC}$

3 $\tan C = \dfrac{\text{Opposite}}{\text{Adjacent}} = \dfrac{AB}{BC}$

For example:

In the opposite figure :



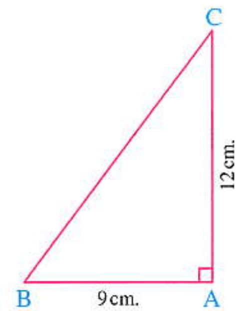
If Δ ABC is a right-angled triangle at B ,

AB = 3 cm. , BC = 4 cm. and AC = 5 cm. , then :

1 sin A = $\frac{4}{5}$	1 sin C = $\frac{3}{5}$
2 cos A = $\frac{3}{5}$	2 cos C = $\frac{4}{5}$
3 tan A = $\frac{4}{3}$	3 tan C = $\frac{3}{4}$

Example 3

In the opposite figure :



Δ ABC is right-angled at A where

AB = 9 cm. and AC = 12 cm.

1 **Find each of :** sin B , cos B , tan B , sin C , cos C and tan C

2 **Prove that :** sin B cos C + cos B sin C = 1

Solution

$\because$ In Δ ABC : m (∠ A) = 90°

$\therefore (BC)^2 = (AB)^2 + (AC)^2$ (Pythagoras' theorem)

$\therefore (BC)^2 = 81 + 144 = 225$ $\qquad \therefore$ BC = 15 cm.

1 $\sin B = \frac{AC}{BC} = \frac{12}{15} = \frac{4}{5}$,

$\cos B = \frac{AB}{BC} = \frac{9}{15} = \frac{3}{5}$,

$\tan B = \frac{AC}{AB} = \frac{12}{9} = \frac{4}{3}$,

$\sin C = \frac{AB}{BC} = \frac{9}{15} = \frac{3}{5}$,

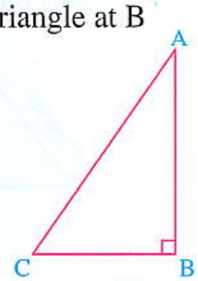
$\cos C = \frac{AC}{BC} = \frac{12}{15} = \frac{4}{5}$,

$\tan C = \frac{AB}{AC} = \frac{9}{12} = \frac{3}{4}$

Remember Pythagoras' theorem :

If ABC is a right-angled triangle at B , then :

- $(AC)^2 = (AB)^2 + (BC)^2$
- $(AB)^2 = (AC)^2 - (BC)^2$
- $(BC)^2 = (AC)^2 - (AB)^2$



2 $\sin B \cos C + \cos B \sin C = \frac{4}{5} \times \frac{4}{5} + \frac{3}{5} \times \frac{3}{5} = \frac{16}{25} + \frac{9}{25} = \frac{25}{25} = 1$

TRY by yourself 2

XYZ is a right-angled triangle at Y , XY = 4 cm. and XZ = 5 cm.

1 ***Find the value of :*** 2 sin X cos X

2 ***Prove that :*** sin X cos Z + cos X sin Z = 1

! Remarks

In the previous example, note that :

1. $\sin B = \cos C = \frac{4}{5}$, $\sin C = \cos B = \frac{3}{5}$

   and by noticing : m (∠ B) + m (∠ C) = 90° "Complementary angles"

We can deduce that :

The **sine** of any acute angle equals the **cosine** of its complementary angle

i.e. If m (∠ A) + m (∠ B) = 90°

, then $\sin A = \cos B$, $\sin B = \cos A$

and vice versa

i.e. If ∠ A and ∠ B are acute angles and sin A = cos B

then m (∠ A) + m (∠ B) = 90°

2. $\frac{\sin B}{\cos B} = \frac{\frac{4}{5}}{\frac{3}{5}} = \frac{4}{3}$, $\tan B = \frac{4}{3}$ $\therefore$ $\tan B = \frac{\sin B}{\cos B}$

   , $\frac{\sin C}{\cos C} = \frac{\frac{3}{5}}{\frac{4}{5}} = \frac{3}{4}$, $\tan C = \frac{3}{4}$ $\therefore$ $\tan C = \frac{\sin C}{\cos C}$

Generally : The tangent of the angle = $\frac{\text{The sine of the angle}}{\text{The cosine of the angle}}$

Example 4

Choose the correct answer from the given ones :

1. If sin 30° = cos θ where θ is the measure of an acute angle

   , then θ = ...

   (a) 15° (b) 30° (c) 60° (d) 90°

2. If x and y are the measures of two complementary angles and

   $\cos x = \frac{4}{5}$, then sin y = ...

   (a) $\frac{3}{5}$ (b) $\frac{4}{5}$ (c) $\frac{3}{4}$ (d) $\frac{4}{3}$

3 In Δ ABC , if m ($\angle$ A) = 60° and sin B = cos B

, then m ($\angle$ C) = ...

(a) 30° (b) 75° (c) 90° (d) 105°

4 If Δ ABC is right-angled at B , then sin A + 2 cos C = ...

(a) 2 sin C (b) 3 sin A (c) 2 sin A (d) 3 cos A

Solution

1 **(c) The reason :** $\because$ sin 30° = cos θ $\therefore$ 30° + θ = 90°

$\therefore$ θ = 60°

2 **(b) The reason :** $\because$ x and y are the measures of two complementary angles

$\therefore$ sin y = cos x $\therefore$ sin y = $\frac{4}{5}$

3 **(b) The reason :** $\because$ sin B = cos B $\therefore$ m ($\angle$ B) = 45°

$\therefore$ m ($\angle$ C) = 180° – (45° + 60°) = 75°

4 **(b) The reason :** $\because$ m ($\angle$ B) = 90° $\therefore$ m ($\angle$ A) + m ($\angle$ C) = 90°

$\therefore$ sin A = cos C

$\therefore$ sin A + 2 cos C = sin A + 2 sin A = 3 sin A

TRY by yourself 3

Choose the correct answer from the given ones :

1 If m ($\angle$ A) = 75° , sin B = cos A where B is an acute angle

, then m ($\angle$ B) = ...

(a) 15° (b) 45° (c) 75° (d) 105°

2 In Δ ABC , if m ($\angle$ B) = 90° , then cos A + sin C = ...

(a) 2 cos C (b) 2 cos A (c) 2 sin A (d) tan A

Example 5

ABC is a triangle in which : AB = AC = 10 cm. , BC = 12 cm. ,

$\overrightarrow{AD}$ is drawn perpendicular to $\overline{BC}$ to cut it at D

1 **Find the value of :** sin B + cos C

2 **Find the value of :** tan ($\angle$ CAD)

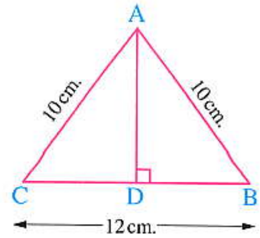
3 **Show that :** sin C + cos C > 1 **and find the value of :** $\sin^2$ C + $\cos^2$ C

and deduce that : $\sin^2$ C + $\cos^2$ C < sin C + cos C

Solution $\because \overline{AD} \perp \overline{BC}$ and $AB = AC$

$\therefore$ D is the midpoint of $\overline{BC}$

$\therefore BD = DC = 6$ cm.



In Δ ADB :

$\because m(\angle ADB) = 90°$

$\therefore (AD)^2 = (AB)^2 - (BD)^2$ (Pythagoras' theorem)

$\therefore (AD)^2 = 100 - 36 = 64$ $\qquad \therefore AD = 8$ cm.

1 $\because \sin B = \frac{AD}{AB} = \frac{8}{10} = \frac{4}{5}$, $\cos C = \frac{CD}{AC} = \frac{6}{10} = \frac{3}{5}$

$\therefore \sin B + \cos C = \frac{4}{5} + \frac{3}{5} = \frac{7}{5}$

2 $\tan(\angle CAD) = \frac{CD}{AD} = \frac{6}{8} = \frac{3}{4}$

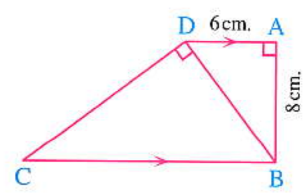
3 $\because \sin C = \frac{AD}{AC} = \frac{8}{10} = \frac{4}{5}$, $\cos C = \frac{3}{5}$

$\therefore \sin C + \cos C = \frac{4}{5} + \frac{3}{5} = \frac{7}{5}$ $\qquad \therefore \sin C + \cos C > 1$

, $\sin^2 C + \cos^2 C = \left(\frac{4}{5}\right)^2 + \left(\frac{3}{5}\right)^2 = \frac{16}{25} + \frac{9}{25} = 1$

$\therefore \sin^2 C + \cos^2 C < \sin C + \cos C$

Example 6 **In the opposite figure :**



ABCD is a quadrilateral in which :

$m(\angle A) = m(\angle BDC) = 90°$

, $\overline{AD} \parallel \overline{BC}$, $AD = 6$ cm. and $AB = 8$ cm.

Find the length of $\overline{DC}$

Solution In Δ ABD :

$\because m(\angle A) = 90°$

$\therefore (DB)^2 = (AB)^2 + (AD)^2 = 64 + 36 = 100$

$\therefore DB = 10$ cm.

, $\because \overline{AD} \parallel \overline{BC}$ and $\overleftrightarrow{BD}$ is a transversal

$\therefore$ m ($\angle$ ADB) = m ($\angle$ DBC) "Alternate angles"

$\therefore$ tan ($\angle$ ADB) = tan ($\angle$ DBC)

$\therefore \frac{AB}{AD} = \frac{DC}{BD}$ $\qquad$ $\therefore \frac{8}{6} = \frac{DC}{10}$

$\therefore DC = \frac{10 \times 8}{6} = 13\frac{1}{3}$ cm. (The req.)

Notice that : ***Also , you can solve this example by using the similarity.***

TRY by yourself 4

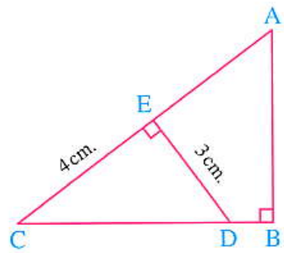
In the opposite figure :

ABC is a triangle in which :

m ($\angle$ B) = 90° , D $\in \overline{BC}$, E $\in \overline{AC}$

where $\overline{DE} \perp \overline{AC}$, DE = 3 cm. and EC = 4 cm.

Prove that : sin A cos C + sin C cos ($\angle$ EDC) = 1

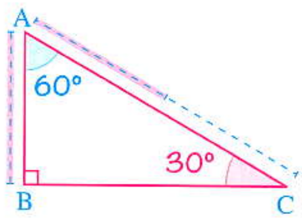


Lesson **2**

The main trigonometrical ratios of some angles

The main trigonometrical ratios of the angles measuring 30° and 60°

In the opposite figure :



ABC is a right-angled triangle at B in which : m (∠ A) = 60° and m (∠ C) = 30° and it is called "thirty and sixty triangle".

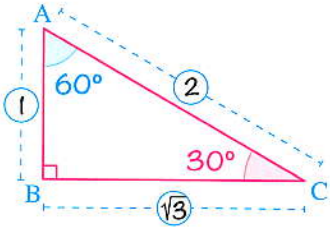
And in it , the length of the side opposite to the angle of measure 30° equals half the length of the hypotenuse.

i.e. $AB = \frac{1}{2} AC$

Assume that : The length of $\overline{AB} = \ell$ length unit , then the length of $\overline{AC} = 2\ell$ length unit.

By applying Pythagoras' theorem to find the length of $\overline{BC}$ **, we find that :**

$$BC = \sqrt{(AC)^2 - (AB)^2} = \sqrt{4\ell^2 - \ell^2} = \sqrt{3\ell^2} = \sqrt{3}\,\ell \text{ length unit.}$$



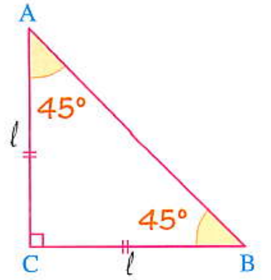
i.e. $AB : AC : BC = \ell : 2\ell : \sqrt{3}\ell = 1 : 2 : \sqrt{3}$

And from Δ ABC , we can find the main trigonometrical ratios of the angles measuring 30° and 60° as follows :

30° $\sin 30° = \frac{AB}{AC} = \frac{1}{2}$	$\cos 30° = \frac{BC}{AC} = \frac{\sqrt{3}}{2}$	$\tan 30° = \frac{AB}{BC} = \frac{1}{\sqrt{3}}$
60° $\sin 60° = \frac{BC}{AC} = \frac{\sqrt{3}}{2}$	$\cos 60° = \frac{AB}{AC} = \frac{1}{2}$	$\tan 60° = \frac{BC}{AB} = \sqrt{3}$

The main trigonometrical ratios of the angle measuring 45°

In the opposite figure :



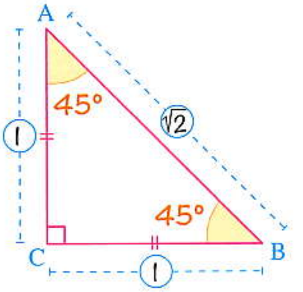
ABC is an isosceles triangle where AC = BC = ℓ length unit and m (∠ C) = 90° ∴ m (∠ A) = m (∠ B) = 45°

By applying Pythagoras' theorem to find the length of $\overline{AB}$

, we find that : $AB = \sqrt{(AC)^2 + (BC)^2}$

$= \sqrt{\ell^2 + \ell^2} = \sqrt{2\ell^2} = \sqrt{2}\,\ell$ length unit.

i.e. $AC : BC : AB = \ell : \ell : \sqrt{2}\,\ell = 1 : 1 : \sqrt{2}$



From Δ ABC , we can find the main trigonometrical ratios of the angle measuring 45° as follows :

45°		
$\sin 45° = \frac{1}{\sqrt{2}}$	$\cos 45° = \frac{1}{\sqrt{2}}$	$\tan 45° = 1$

∗ **And the following table summarizes the main trigonometrical ratios of the angles whose measures are 30° , 60° and 45° :**

The measure of the angle / The trigonometrical ratio	30°	60°	45°
sin	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$
cos	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$
tan	$\frac{1}{\sqrt{3}}$	$\sqrt{3}$	1

Example 1 **Find the value of :** $\sin 30° \cos 60° + \cos^2 30° + 5 \tan 45° - 10 \cos^2 45°$

Solution

The expression $= \frac{1}{2} \times \frac{1}{2} + \left(\frac{\sqrt{3}}{2}\right)^2 + 5 \times 1 - 10 \times \left(\frac{1}{\sqrt{2}}\right)^2$

$= \frac{1}{4} + \frac{3}{4} + 5 - \frac{10}{2} = 1 + 5 - 5 = 1$

Example 2 **Prove that :** $\sin^2 60° + \sin^2 45° + \sin^2 30° = \cos^2 30° + \frac{1}{3} \tan^2 60° - \cos^2 60°$

Solution

L.H.S. $= \left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{2}\right)^2 = \frac{3}{4} + \frac{1}{2} + \frac{1}{4} = \frac{3}{2}$

R.H.S. $= \left(\frac{\sqrt{3}}{2}\right)^2 + \frac{1}{3}\left(\sqrt{3}\right)^2 - \left(\frac{1}{2}\right)^2 = \frac{3}{4} + 1 - \frac{1}{4} = \frac{3}{2}$

∴ The two sides are equal.

TRY by yourself 1

1 ***Find the value of :*** **(1)** $\cos 60° + \sin 30° - \tan 45°$ **(2)** $\sin^2 30° + \sin^2 60°$

2 ***Prove that :*** $2 \sin 30° + 4 \cos 60° = \tan^2 60°$

Example 3

Find the value of X which satisfies :

1 $X \sin 30° \cos^2 45° = \cos^2 30°$

2 $2 \sin X = \tan^2 60° - 2 \tan 45°$ where X is the measure of an acute angle.

Solution

1 $\because X \sin 30° \cos^2 45° = \cos^2 30°$ $\therefore X \times \frac{1}{2} \times \left(\frac{1}{\sqrt{2}}\right)^2 = \left(\frac{\sqrt{3}}{2}\right)^2$

$\therefore \frac{1}{4} X = \frac{3}{4}$ $\therefore X = 3$

2 $\because 2 \sin X = \tan^2 60° - 2 \tan 45°$ $\therefore 2 \sin X = \left(\sqrt{3}\right)^2 - 2 \times 1 = 3 - 2 = 1$

$\therefore \sin X = \frac{1}{2}$ $\therefore X = 30°$

TRY by yourself 2

Find the value of X which satisfies :

1 $X \cos 30° = \tan 60°$

2 $\tan X = \frac{2 \tan 30°}{1 - \tan^2 30°}$ where X is the measure of an acute angle.

Example 4

Choose the correct answer from the given ones :

1 If $\cos 4 X = \frac{1}{2}$ where X is the measure of an acute angle , then $X =$ ....

(a) 15° (b) 30° (c) 45° (d) 60°

2 If $\tan (X + 10°) = \sqrt{3}$ where $(X + 10°)$ is the measure of an acute angle , then $X =$ ....

(a) 20° (b) 40° (c) 50° (d) 70°

3 If $\sin X = \frac{1}{2}$ where X is the measure of an acute angle , then $\sin 2 X =$ ....

(a) 1 (b) $\frac{1}{4}$ (c) $\frac{\sqrt{3}}{2}$ (d) $\frac{1}{\sqrt{3}}$

4 If $\cos (X + 15°) = \frac{1}{2}$ where $(X + 15°)$ is the measure of an acute angle , then $\sin (75° - X) =$ ....

(a) $\frac{1}{2}$ (b) $\frac{\sqrt{3}}{2}$ (c) $\frac{1}{\sqrt{2}}$ (d) 1

5 If $4 \cos 60° \sin 30° = \tan X$ where X is the measure of an acute angle , then $X =$ ....

(a) 30° (b) 45° (c) 60° (d) 90°

Solution

1 **(a) The reason :** $\because \cos 4X = \frac{1}{2}$ $\therefore 4X = 60°$

$\therefore X = \frac{60°}{4} = 15°$

2 **(c) The reason :** $\because \tan (X + 10°) = \sqrt{3}$ $\therefore X + 10° = 60°$

$\therefore X = 60° - 10° = 50°$

3 **(c) The reason :** $\because \sin X = \frac{1}{2}$ $\therefore X = 30°$

$\therefore \sin 2X = \sin 60° = \frac{\sqrt{3}}{2}$

4 **(a) The reason :** $\because \cos (X + 15°) = \frac{1}{2}$ $\therefore X + 15° = 60°$

$\therefore X = 60° - 15° = 45°$

$\therefore \sin (75° - X) = \sin (75° - 45°) = \sin 30° = \frac{1}{2}$

5 **(b) The reason :** $\because 4 \cos 60° \sin 30° = 4 \times \frac{1}{2} \times \frac{1}{2} = 1$

$\therefore \tan X = 1$ $\therefore X = 45°$

TRY by yourself 3

Choose the correct answer from the given ones :

1 $2 \cos^2 30° - 1 =$ ....

(a) cos 60° (b) sin 60°

(c) 2 sin 30° (d) tan 60°

2 If $\tan (X + 15°) = 1$ where $(X + 15°)$ is the measure of an acute angle , then $X =$ ....

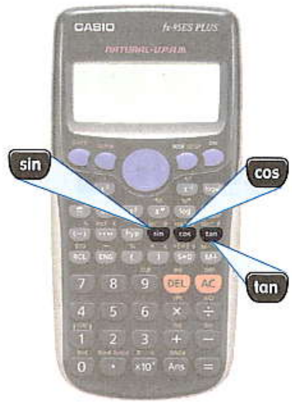
(a) 15° (b) 30° (c) 45° (d) 60°

3 If $\left(\cos X , \frac{1}{2}\right) = \left(\frac{1}{2} , \sin y\right)$ where X and y are the measures of two acute angles , then $X + y =$ ....

(a) 30° (b) 60° (c) 90° (d) 120°

Using the calculator

First Finding the main trigonometrical ratios of a given angle



In the calculator , there are three keys : [sin] , [cos] , [tan]

1. The key [sin] means sine.
2. The key [cos] means cosine.
3. The key [tan] means tangent.

By using these keys we can find the main trigonometrical ratios of any angle if its measure is known.

Example 5 **By using the calculator , find the value of each of the following approximated to the nearest four decimals :**

1 $\sin 36^\circ$ 2 $\cos 72^\circ\ 35'$ 3 $\tan 50^\circ\ 46'\ 25''$

Solution Use the keys of the calculator as the following sequence from left :

1 [sin] [3] [6] [=]

$\therefore \sin 36^\circ \simeq 0.5878$

2 [cos] [7] [2] [°'''] [3] [5] [°'''] [=]

$\therefore \cos 72^\circ\ 35' \simeq 0.2993$

3 [tan] [5] [0] [°'''] [4] [6] [°'''] [2] [5] [°'''] [=]

$\therefore \tan 50^\circ\ 46'\ 25'' \simeq 1.2250$

TRY by yourself 4 ***By using the calculator, find the value of each of the following approximated to the nearest three decimals :***

1 $\sin 35^\circ\ 12'$ 2 $\tan 58^\circ\ 24'$

Second: Finding the measure of an angle if one of its trigonometrical ratios is given

If sin A = 0.6218 , then A is the measure of the angle whose sine is 0.6218

To find the measure of this angle , we can use the calculator as the following sequence from left :

SHIFT sin . 6 2 1 8 = °''' Then $A \simeq 38^\circ\ 26'\ 52''$

Example 6

Find A in each of the following , where A is the measure of an acute angle :

1 sin A = 0.8 **2** cos A = 0.7152 **3** tan A = 1.5156

Solution

Use the keys of the calculator as the following sequence from left :

1 SHIFT sin . 8 = °'''

$\therefore A \simeq 53^\circ\ 7'\ 48''$

2 SHIFT cos . 7 1 5 2 = °'''

$\therefore A \simeq 44^\circ\ 20'\ 25''$

3 SHIFT tan 1 . 5 1 5 6 = °'''

$\therefore A \simeq 56^\circ\ 34'\ 59''$

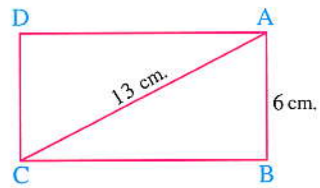
TRY by yourself 5

Using the calculator , find A in each of the following where A is the measure of an acute angle :

1 sin A = 0.3945 **2** cos A = 0.3824

Example 7

In the opposite figure :



ABCD is a rectangle in which :

AB = 6 cm. and AC = 13 cm. **Find :**

1 m (∠ ACB)

2 The area of the rectangle ABCD to the nearest one decimal digit.

Solution

$\because$ ABCD is a rectangle.

$\therefore$ m ($\angle$ B) = 90°

In Δ ABC :

$\sin (\angle ACB) = \frac{AB}{AC} = \frac{6}{13}$

And by using the calculator :

$\therefore$ m ($\angle$ ACB) $\simeq 27° \, 29' \, 11''$ (First req.)

, $\because \cos (\angle ACB) = \frac{BC}{AC}$

$\therefore \cos 27° \, 29' \, 11'' = \frac{BC}{13}$

$\therefore BC = 13 \times \cos 27° \, 29' \, 11''$

$\therefore$ The area of the rectangle ABCD = AB × BC

$= 6 \times 13 \times \cos 27° \, 29' \, 11'' \simeq 69.2 \text{ cm}^2$.

(Second req.)

Notice that :

Also , you can find the length of $\overline{BC}$ by using Pythagoras' theorem in Δ ABC

TRY by yourself 6

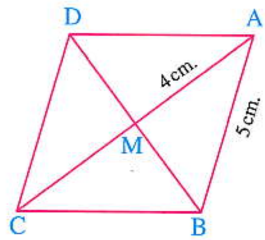
In the opposite figure :

ABCD is a rhombus , whose diagonals intersect at M

If AB = 5 cm. and AM = 4 cm.

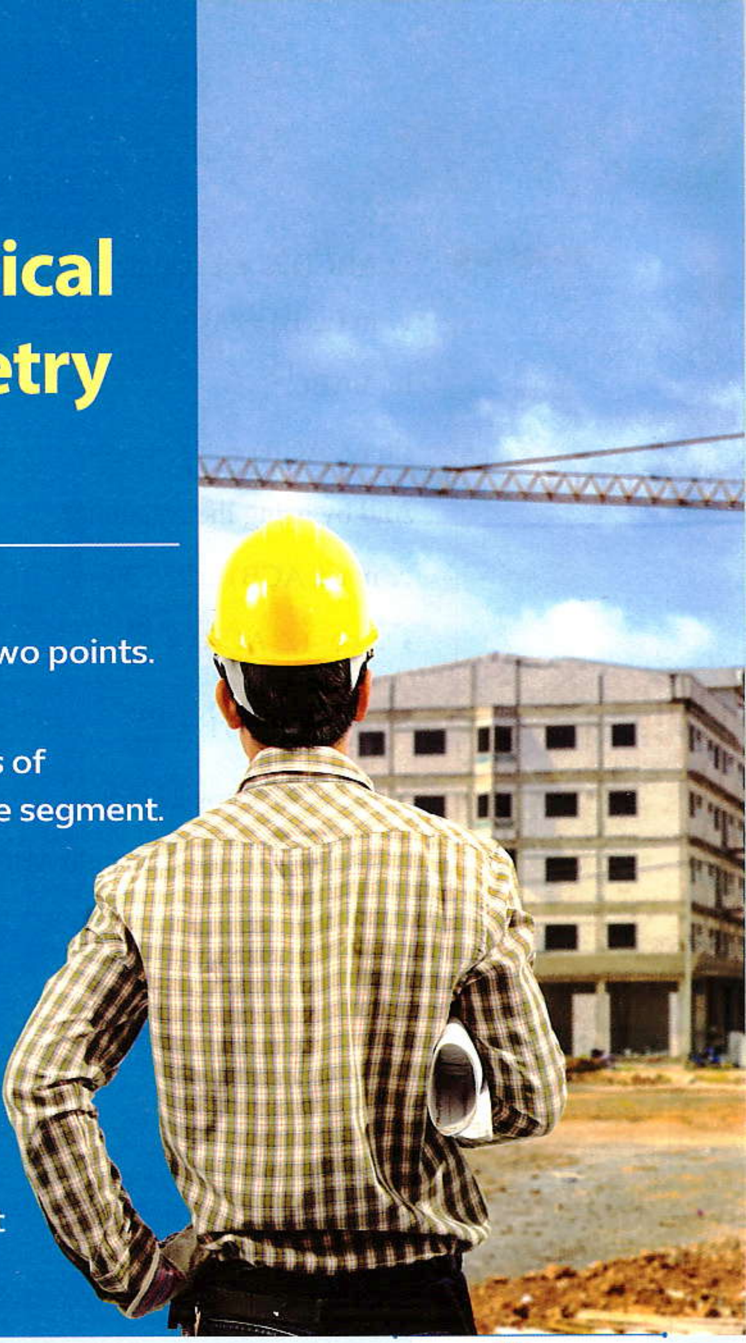
, ***find :***

1. m ($\angle$ BAD)
2. The area of the rhombus ABCD



UNIT **5**

Analytical geometry



Lesson One Distance between two points.

Lesson Two The two coordinates of the midpoint of a line segment.

Lesson Three The slope of the straight line.

Lesson Four The equation of the straight line given its slope and the intercepted part of y-axis.

Unit Objectives : By the end of this unit, student should be able to :

- find the distance between two points in the coordinates plane.
- find the two coordinates of the midpoint of a line segment.
- recognize the slope of the straight line.
- find the slope of the straight line given the measure of the positive angle which this straight line makes with the positive direction of the x-axis.
- recognize the relation between the two slopes of two parallel straight lines.
- recognize the relation between the two slopes of two perpendicular straight lines.
- find the slope of the straight line and the length of the intercepted part from y-axis given the equation of the straight line.
- find the equation of the straight line given its slope and the length of the intercepted part from y-axis.
- use the slope of the straight line for solving some life problems.

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Lesson **1**

Distance between two points

Let M (x_1 , y_1) and N (x_2 , y_2) be two points in the same coordinates plane.

From the geometry of the figure we find that :

NL = NB – LB = $y_2 - y_1$

Generally NL = $|y_2 - y_1|$

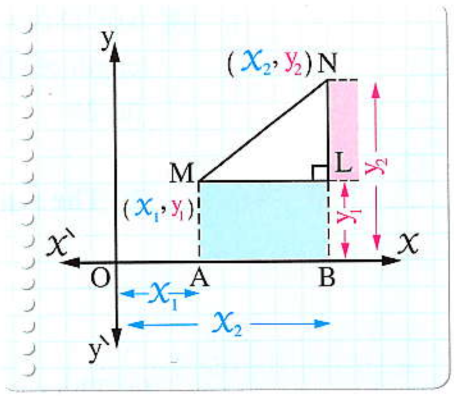
Similarly LM = BO – AO = $x_2 - x_1$

Generally LM = $|x_2 - x_1|$

$\because$ Δ NLM is right-angled at L

$\therefore (MN)^2 = (LM)^2 + (NL)^2$

$\therefore (MN)^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$



$$\therefore MN = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

i.e.

The distance between the two points M and N equals $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

and we know that :

$(x_2 - x_1)^2 = (x_1 - x_2)^2$, and similarly : $(y_2 - y_1)^2 = (y_1 - y_2)^2$, therefore :

The distance between the two points M and N equals also $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

Generally :

The distance between two points =

$\sqrt{\text{square of the difference between } x\text{-coordinates} + \text{square of the difference between y-coordinates}}$

For example : If A (3 , 6) and B (– 1 , 4) , then

the length of $\overline{AB} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(-1-3)^2 + (4-6)^2} = \sqrt{(-4)^2 + (-2)^2}$

$= \sqrt{16+4} = \sqrt{20} = 2\sqrt{5}$ length unit.

you can find the length of $\overline{AB}$ as follows : the length of $\overline{AB}$

$= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = \sqrt{(3-(-1))^2 + (6-4)^2} = \sqrt{4^2 + 2^2} = \sqrt{16+4} = \sqrt{20} = 2\sqrt{5}$ length unit.

Example 1

Choose the correct answer from the given ones :

1. The distance between the two points (6 , 0) and (0 , 8) equals .... length unit.
   (a) 12 (b) 10 (c) 8 (d) 6
2. The distance between the point A $(\sqrt{2}, 4)$ and the origin point equals .... length unit.
   (a) $\sqrt{2}$ (b) $2\sqrt{2}$ (c) $3\sqrt{2}$ (d) $4\sqrt{2}$
3. The distance between the point (– 7 , – 3) and y-axis equals .... length unit.
   (a) – 7 (b) – 3 (c) 7 (d) 3
4. ABCD is a rectangle in which A (– 1 , – 3) and C (2 , 1) , then the length of $\overline{BD}$ = ... length unit.
   (a) 25 (b) 5 (c) $\sqrt{7}$ (d) $\sqrt{5}$

Solution

1. **(b) The reason :** The required distance $= \sqrt{(0-6)^2 + (8-0)^2}$
   $= \sqrt{(-6)^2 + (8)^2} = \sqrt{36+64}$
   $= \sqrt{100} = 10$ length unit.
2. **(c) The reason :** The distance between any point (x , y) and the origin point (0 , 0) equals $\sqrt{x^2 + y^2}$
   ∴ The required distance $= \sqrt{(\sqrt{2})^2 + (4)^2}$
   $= \sqrt{2+16} = \sqrt{18} = \sqrt{9 \times 2}$
   $= 3\sqrt{2}$ length unit.
3. **(c) The reason :** The distance between the point (– 7 , – 3) and $\overleftrightarrow{yy}$ equals | – 7 | because the distance is a positive number.
   ∴ The required distance = 7 length unit.

4 **(b) The reason :** The length of $\overline{BD}$ = the length of $\overline{AC}$ because the rectangle diagonals are equal in length.

$\therefore$ The length of $\overline{BD} = \sqrt{(2+1)^2 + (1+3)^2}$

$= \sqrt{3^2 + 4^2} = \sqrt{9 + 16}$

$= \sqrt{25} = 5$ length unit.

Example 2 If the distance between the two points (a , 5) and (3 a – 1 , 1) equals 5 length units , **find the value of :** a

Solution

$\because \sqrt{(3a - 1 - a)^2 + (1 - 5)^2} = 5$

$\therefore \sqrt{(2a - 1)^2 + (-4)^2} = 5$ "Squaring the two sides"

$\therefore (2a - 1)^2 + 16 = 25$

$\therefore (2a - 1)^2 = 9$ "Taking the square root of the two sides"

$\therefore 2a - 1 = \pm 3$

$\therefore 2a - 1 = 3$ thus , $2a = 4$ $\therefore$ $a = 2$

or $2a - 1 = -3$ thus , $2a = -2$ $\therefore$ $a = -1$

TRY by yourself 1 If A (2 , 5) and B (– 1 , 1) , ***find the length of :*** $\overline{AB}$

Example 3 If ABC is a triangle where A (0 , 0) , B (3 , 4) and C (– 4 , 3) , find the perimeter of Δ ABC

Solution

$\because$ The perimeter of Δ ABC = AB + BC + CA

, $AB = \sqrt{3^2 + 4^2}$

$= \sqrt{9 + 16} = \sqrt{25} = 5$ length unit.

, $BC = \sqrt{(-4 - 3)^2 + (3 - 4)^2}$

$= \sqrt{(-7)^2 + (-1)^2}$

$= \sqrt{49 + 1} = \sqrt{50} = 5\sqrt{2}$ length unit.

, $CA = \sqrt{(-4)^2 + (3)^2} = \sqrt{16 + 9} = \sqrt{25} = 5$ length unit.

$\therefore$ The perimeter of Δ ABC $= 5 + 5\sqrt{2} + 5 = \left(10 + 5\sqrt{2}\right)$ length unit.

Example 4 **Prove that :** ΔABC is an equilateral triangle where : A (6 , 0) , B (2 , 0) and $C\left(4 , 2\sqrt{3}\right)$, then find its area.

Solution $\because AB = \sqrt{(6-2)^2+(0-0)^2} = \sqrt{16+0} = \sqrt{16} = 4$ length unit.

, $BC = \sqrt{(2-4)^2+(0-2\sqrt{3})^2} = \sqrt{4+12} = \sqrt{16} = 4$ length unit.

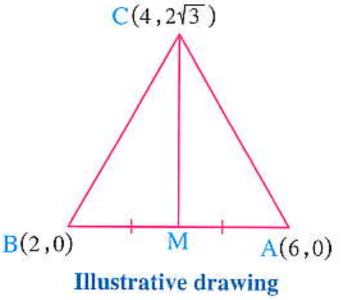
and $AC = \sqrt{(6-4)^2+(0-2\sqrt{3})^2}$

$= \sqrt{4+12} = \sqrt{16} = 4$ length unit.

$\therefore AB = BC = AC$ $\therefore \Delta ABC$ is equilateral

Let M be the midpoint of the base $\overline{AB}$

$\therefore \overline{CM} \perp \overline{AB}$



Illustrative drawing

$\therefore$ By using Pythagoras' theorem , we find that :

$\therefore$ The height $MC = \sqrt{(AC)^2-(AM)^2} = \sqrt{4^2-2^2} = \sqrt{12} = 2\sqrt{3}$ length unit

$\therefore$ The area of $\Delta ABC = \frac{1}{2} \times AB \times MC = \frac{1}{2} \times 4 \times 2\sqrt{3} = 4\sqrt{3}$ square unit.

TRY by yourself 2

Prove that : ΔABC is an isosceles triangle where : A (3 , 3) , B (5 , 9) and C (– 1 , 7)

! Remark 1

To prove that three given points are collinear (**i.e.** They lie on one straight line) we can find the distance between each two of these points , then prove that the greatest distance equals the sum of the two other distances.

Example 5 **Prove that :** The points A (– 2 , 7) , B (– 3 , 4) and C (1 , 16) are collinear.

Solution $\because AB = \sqrt{(-2+3)^2+(7-4)^2} = \sqrt{1+9} = \sqrt{10}$ length unit.

, $BC = \sqrt{(-3-1)^2+(4-16)^2} = \sqrt{16+144} = \sqrt{160} = 4\sqrt{10}$ length unit.

and $AC = \sqrt{(-2-1)^2+(7-16)^2} = \sqrt{9+81} = \sqrt{90} = 3\sqrt{10}$ length unit.

$\therefore BC = AB + AC$ $\therefore$ A , B and C are collinear.

Remark 2

- To prove that the points A , B and C are the vertices of a triangle , we can find AB , BC and AC , then prove that the sum of the smaller two lengths is greater than the third length.
- To determine the type of the triangle ABC according to its angle measures (where $\overline{AC}$ is the longest side of the triangle ABC)
  , we compare between $(AC)^2$ and $(AB)^2 + (BC)^2$ as the following :
  1. If $(AC)^2 > (AB)^2 + (BC)^2$, then the triangle is obtuse-angled at B
  2. If $(AC)^2 = (AB)^2 + (BC)^2$, then the triangle is right-angled at B
  3. If $(AC)^2 < (AB)^2 + (BC)^2$, then the triangle is acute-angled.

Example 6 **Prove that :** The triangle whose vertices are A (3 , 2) , B (– 4 , 1) and C (2 , – 1) is right-angled , then find its area.

Solution

$\because AB = \sqrt{(3+4)^2 + (2-1)^2}$

$= \sqrt{49+1} = \sqrt{50}$ length unit.

$, BC = \sqrt{(-4-2)^2 + (1+1)^2}$

$= \sqrt{36+4} = \sqrt{40}$ length unit.

and $AC = \sqrt{(3-2)^2 + (2+1)^2}$

$= \sqrt{1+9} = \sqrt{10}$ length unit.

$\because (AC)^2 + (BC)^2 = 10 + 40 = 50$

$, (AB)^2 = 50$

$\therefore (AC)^2 + (BC)^2 = (AB)^2$

$\therefore \Delta ABC$ is right-angled at C

$\therefore$ The area of the triangle ABC $= \frac{1}{2} AC \times BC$

$= \frac{1}{2} \times \sqrt{10} \times \sqrt{40}$

$= \frac{1}{2} \times \sqrt{10} \times 2\sqrt{10} = 10$ square unit.

TRY by yourself 3

If A (– 1 , – 1) , B (2 , 3) and C (6 , 0)

, ***prove that :*** Δ ABC is right-angled at B , then find its area.

Remark 3

If ABCD is a quadrilateral :

1. To prove that ABCD is a parallelogram , we prove that : AB = CD , BC = AD
2. To prove that ABCD is a rhombus , we prove that : AB = BC = CD = DA
3. To prove that ABCD is a rectangle , we prove that : AB = CD , BC = AD , AC = BD
4. To prove that ABCD is a square , we prove that : AB = BC = CD = DA , AC = BD

Example 7 If A (3 , – 2) , B (– 5 , 0) , C (0 , – 7) and D (8 , – 9) ,

prove that : ABCD is a parallelogram.

Solution

$\because$ AB $= \sqrt{(3+5)^2 + (-2-0)^2} = \sqrt{64+4}$

$= \sqrt{68}$ length unit.

, BC $= \sqrt{(-5-0)^2 + (0+7)^2} = \sqrt{25+49}$

$= \sqrt{74}$ length unit.

, CD $= \sqrt{(0-8)^2 + (-7+9)^2} = \sqrt{64+4}$

$= \sqrt{68}$ length unit.

and DA $= \sqrt{(8-3)^2 + (-9+2)^2} = \sqrt{25+49} = \sqrt{74}$ length unit.

$\therefore$ AB = CD , BC = DA $\quad\therefore$ ABCD is a parallelogram.

Example 8 **Prove that :** The points A (– 1 , 4) , B (1 , 1) , C (– 1 , – 2) and D (– 3 , 1) are the vertices of a rhombus and graph it , then find its area.

Solution

$\because$ AB $= \sqrt{(-1-1)^2 + (4-1)^2}$

$= \sqrt{4+9} = \sqrt{13}$ length unit.

, BC $= \sqrt{(1+1)^2 + (1+2)^2}$

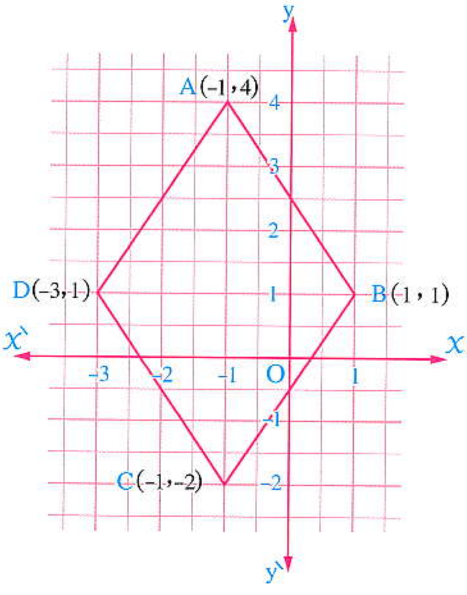
$= \sqrt{4+9} = \sqrt{13}$ length unit.

, CD $= \sqrt{(-1+3)^2 + (-2-1)^2}$

$= \sqrt{4+9} = \sqrt{13}$ length unit.

and DA $= \sqrt{(-3+1)^2 + (1-4)^2}$

$= \sqrt{4+9} = \sqrt{13}$ length unit.



$\therefore$ AB = BC = CD = DA

$\therefore$ The quadrilateral ABCD is a rhombus.

$\because$ $AC = \sqrt{(-1+1)^2 + (4+2)^2} = \sqrt{0+36} = \sqrt{36} = 6$ length unit.

, $BD = \sqrt{(1+3)^2 + (1-1)^2} = \sqrt{16+0} = \sqrt{16} = 4$ length unit.

$\therefore$ The area of the rhombus ABCD = $\frac{1}{2} \times 6 \times 4 = 12$ square unit.

TRY by yourself 4

Prove that : The points A (– 1 , 3) , B (5 , 1) , C (6 , 4) and D (0 , 6) are the vertices of a rectangle , then find its area.

Remark 4

- The axis of symmetry of a line segment is the straight line that is perpendicular to it at its midpoint.
- Any point on the axis of symmetry of a line segment is at equal distances from its terminals.

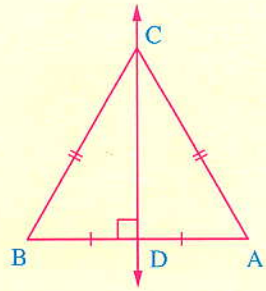
The converse is true , i.e. If a point is at equal distances from the two terminals of a line segment , then this point lies on the axis of this line segment.

For example:

In the opposite figure :

If CA = CB

, then C $\in$ the axis of symmetry of $\overline{AB}$



Example 9 If A (1 , – 1) and B (1 , 3)

, **prove that :** The point C (– 1 , 1) lies on the axis of symmetry of $\overline{AB}$

Solution $\because$ $CA = \sqrt{(-1-1)^2 + (1+1)^2} = \sqrt{4+4} = \sqrt{8} = 2\sqrt{2}$ length unit.

, $CB = \sqrt{(-1-1)^2 + (1-3)^2} = \sqrt{4+4} = \sqrt{8} = 2\sqrt{2}$ length unit.

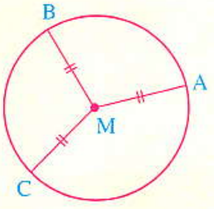
$\therefore$ CA = CB

$\therefore$ The point C lies on the axis of symmetry of $\overline{AB}$

Remark 5

- If A $\in$ the circle M , then the radius length of this circle (r) = MA
- To prove that : Three points as A , B and C lie on the same circle of centre M

  we prove that : MA = MB = MC



- Remember that :

  - The circumference of the circle = $2 \pi r$
  - The area of the circle = πr^2

Example 10

Choose the correct answer from the given ones :

1 The diameter length of the circle of centre A (– 2 , 3) and passing through B (2 , – 1) equals .... length unit.

(a) $8\sqrt{2}$ (b) $4\sqrt{2}$ (c) 5 (d) 4

2 A circle is of centre (3 , – 4) and its radius length is 5 length unit. Which of the following points belongs to this circle ?

(a) (– 3 , 4) (b) (0 , 0) (c) (5 , 0) (d) (0 , 4)

Solution

1 **(a) The reason :** r = the length of $\overline{AB} = \sqrt{(2+2)^2 + (-1-3)^2}$

$= \sqrt{(4)^2 + (-4)^2} = \sqrt{32}$

$= 4\sqrt{2}$ length unit.

$\therefore$ The diameter length $= 2\,r = 2 \times 4\sqrt{2}$

$= 8\sqrt{2}$ length unit.

2 **(b) The reason :** The right answer is the point whose distance from the centre of the circle equals the radius length of the circle. Finding the distance between each point and the centre of the circle (3 , – 4) , you find that (0 , 0) is the right answer because

$\sqrt{(3-0)^2 + (-4-0)^2} = \sqrt{9+16} = \sqrt{25} = 5$ length unit = r

Example 11 **Prove that :** The points A (– 6 , 2) , B (0 , 8) and C (– 8 , 4) lie on the circle whose centre is M (– 4 , 6) and find its area where $\pi \approx 3.14$

Solution $\because$ MA $= \sqrt{(-6+4)^2+(2-6)^2} = \sqrt{4+16} = \sqrt{20} = 2\sqrt{5}$ length units.

, MB $= \sqrt{(0+4)^2+(8-6)^2} = \sqrt{16+4} = \sqrt{20} = 2\sqrt{5}$ length units.

and MC $= \sqrt{(-8+4)^2+(4-6)^2} = \sqrt{16+4} = \sqrt{20} = 2\sqrt{5}$ length units.

$\therefore$ MA = MB = MC

$\therefore$ The points A , B and C lie on the circle M whose radius length $r = 2\sqrt{5}$ length units.

$\therefore$ The area of the circle M $= \pi r^2 \approx 3.14 \times (2\sqrt{5})^2 \approx 62.8$ square units.

TRY by yourself 5

Prove that : The points A (– 2 , 0) , B (5 , 1) and C (6 , – 6) lie on the circle whose centre is M (2 , – 3) and find the circumference of the circle in terms of π

Lesson 2

The two coordinates of the midpoint of a line segment

If A (x_1, y_1) and B (x_2, y_2) are two points in a coordinates plane and M (x, y) is the midpoint of $\overline{AB}$

WATCH VIDEO

From the opposite figure :

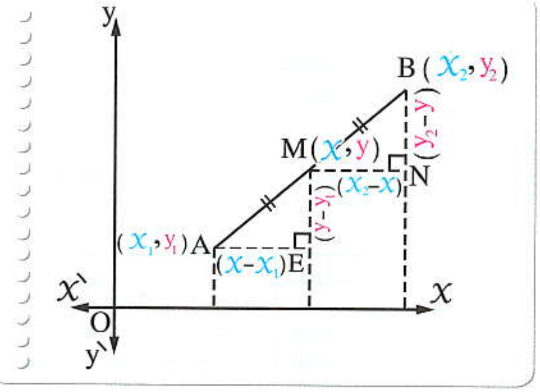
Δ AEM and Δ MNB are congruent

$\therefore$ AE = MN , EM = NB

$\therefore x - x_1 = x_2 - x$, $y - y_1 = y_2 - y$

$\therefore 2x = x_1 + x_2$, $2y = y_1 + y_2$

$\therefore x = \frac{x_1 + x_2}{2}$, $y = \frac{y_1 + y_2}{2}$



$$\therefore M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

For example:

If X (3 , – 2) , Y (– 1 , – 4) and M is the midpoint of $\overline{XY}$, then :

$$M = \left(\frac{3 + (-1)}{2}, \frac{-2 + (-4)}{2}\right) = (1, -3)$$

Example 1 If C (10 , – 4) is the midpoint of $\overline{AB}$ where A (4 , – 2) , find the point B

Solution Let B (x , y)

Notice that : If (a , b) = (c , d) , then a = c , b = d

$\because$ C is the midpoint of $\overline{AB}$

$\therefore (10 , -4) = \left(\frac{x+4}{2} , \frac{y+(-2)}{2}\right)$

$\therefore \frac{x+4}{2} = 10$ $\therefore x + 4 = 20$ $\therefore x = 16$

, $\frac{y-2}{2} = -4$ $\therefore y - 2 = -8$ $\therefore y = -6$ $\therefore$ B = (16 , – 6)

TRY by yourself 1

If C is the midpoint of $\overline{AB}$, then find the value of each of x and y in each of the following :

1. A (2 , 5) , B (– 2 , – 3) and C (x , y)
2. A (x , 4) , B (– 1 , – 6) and C (– 2 , y)

! Remark

If $\overline{AB}$ is a diameter in a circle of centre M , then M is the midpoint of $\overline{AB}$

Example 2 If $\overline{AB}$ is a diameter in the circle M where A (4 , – 1) and B (– 2 , 7) , find the point M , then find the circumference and the area of the circle.

Solution $\because$ $\overline{AB}$ is a diameter in the circle M $\therefore$ M is the midpoint of $\overline{AB}$

$\therefore$ The point M $= \left(\frac{4+(-2)}{2} , \frac{-1+7}{2}\right) = (1 , 3)$

, $\because r = AM = \sqrt{(1-4)^2 + (3+1)^2} = \sqrt{9+16} = \sqrt{25} = 5$ length units.

$\therefore$ The circumference of the circle $= 2\pi r = 2\pi \times 5 = 10\pi$ length units.

, the area of the circle $= \pi r^2 = \pi \times 5^2 = 25\pi$ square units.

Another method to calculate the radius length of the circle :

$\because AB = \sqrt{(-2-4)^2 + (7+1)^2} = \sqrt{36+64} = \sqrt{100} = 10$ length units.

, $\because$ $\overline{AB}$ is a diameter $\therefore r = \frac{1}{2} AB = 5$ length units.

, then complete the solution to find the circumference and the area of the circle.

TRY by yourself 2

If $\overline{AB}$ is a diameter in the circle M where A (4 , 1) and B (– 6 , 3) , then find the point M

Example 3 **Prove that :** The quadrilateral ABCD is a parallelogram where A (4 , 3) , B (0 , 2) , C (– 2 , – 3) and D (2 , – 2)

Solution

$\because$ The two diagonals of the quadrilateral are $\overline{AC}$ and $\overline{BD}$

, the midpoint of $\overline{AC} = \left(\frac{4 + (-2)}{2} , \frac{3 + (-3)}{2}\right) = (1 , 0)$

and the midpoint of $\overline{BD} = \left(\frac{0 + 2}{2} , \frac{2 + (-2)}{2}\right) = (1 , 0)$

$\therefore$ The midpoint of $\overline{AC}$ is the same midpoint of $\overline{BD}$

$\therefore$ The two diagonals bisect each other.

$\therefore$ ABCD is a parallelogram.

Notice that :
You can solve this example by using the distance between two points as the previous.

Example 4 **Prove that :** The points A (5 , 1) , B (1 , – 3) and C (– 5 , 3) are the vertices of a right-angled triangle at B , then find the point D that makes the figure ABCD a rectangle.

Solution

$\because AB = \sqrt{(1-5)^2 + (-3-1)^2} = \sqrt{16 + 16} = \sqrt{32}$ length unit.

$, BC = \sqrt{(-5-1)^2 + (3+3)^2} = \sqrt{36 + 36} = \sqrt{72}$ length unit.

$, AC = \sqrt{(-5-5)^2 + (3-1)^2} = \sqrt{100 + 4} = \sqrt{104}$ length unit.

$\therefore (AB)^2 + (BC)^2 = 32 + 72 = 104 = (AC)^2$

$\therefore \Delta$ ABC is a right-angled triangle at B

Let D (x , y) such that the figure ABCD is a rectangle.

$\therefore \overline{AC}$ and $\overline{BD}$ bisect each other.

$\therefore$ The midpoint of $\overline{AC}$ = the midpoint of $\overline{BD}$

$, \because$ the midpoint of $\overline{AC} = \left(\frac{5 - 5}{2} , \frac{1 + 3}{2}\right) = (0 , 2)$

, the midpoint of $\overline{BD} = \left(\frac{x + 1}{2} , \frac{y - 3}{2}\right)$

$\therefore \left(\frac{x + 1}{2} , \frac{y - 3}{2}\right) = (0 , 2)$

$\therefore \frac{x+1}{2}=0$ $\therefore x+1=0$ $\therefore x=-1$

, $\frac{y-3}{2}=2$ $\therefore y-3=4$ $\therefore y=7$

$\therefore D=(-1\ ,7)$

Example 5

Prove that : The triangle whose vertices are A (– 1 , 4) , B (3 , 1) and C (– 5 , 1) is an isosceles triangle , then find its area.

Solution

$\because AB=\sqrt{(3+1)^2+(1-4)^2}=\sqrt{16+9}$

$=5$ length unit.

, $BC=\sqrt{(3+5)^2+(1-1)^2}=\sqrt{64}=8$ length unit.

, $AC=\sqrt{(-5+1)^2+(1-4)^2}=\sqrt{16+9}$

$=5$ length unit.

$\therefore AB=AC$

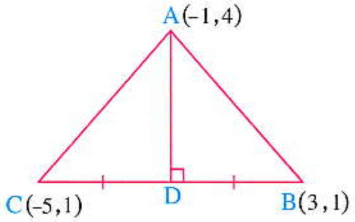
$\therefore \Delta ABC$ is an isosceles triangle.

Let D (x , y) be the midpoint of $\overline{BC}$

$\therefore D=\left(\frac{3-5}{2}, \frac{1+1}{2}\right)=(-1\ ,1)$

, $\because$ D is the midpoint of $\overline{BC}$

$\therefore \overline{AD}\perp\overline{BC}$



Illustrative drawing

, $\because AD=\sqrt{(-1+1)^2+(1-4)^2}=\sqrt{9}=3$ length unit.

, BC = 8 length unit

$\therefore$ The area of $\Delta ABC=\frac{1}{2}BC\times AD=\frac{1}{2}\times 8\times 3=12$ square unit.

TRY by yourself 3

If C is the midpoint of $\overline{AB}$ where A (2 , 3) , B (4 , – 7) and C is the midpoint of $\overline{DE}$ where D (– 3 , 5) , find the point E

Lesson **3**

The slope of the straight line

You studied before the slope of the straight line given two points on it.

If A and B are two points in the coordinates plane where A (x_1, y_1) and B (x_2, y_2)

, then : The slope of the straight line $\overleftrightarrow{AB} = \frac{y_2 - y_1}{x_2 - x_1}$

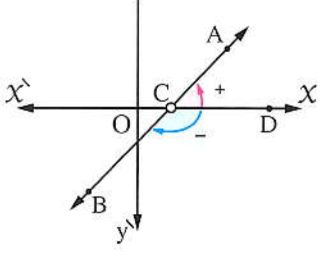
In this lesson , you will learn :

- How to find the slope of the straight line given the measure of the positive angle which this straight line makes with the positive direction of the x-axis.
- The relation between the slopes of two parallel straight lines.
- The relation between the slopes of two perpendicular straight lines.

And before studying these topics , you will study the positive and negative measures of an angle.

The positive measure and the negative measure of an angle

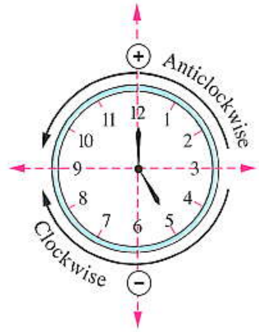
In the opposite figure :



If $\overleftrightarrow{AB}$ intersects the x-axis at the point C , then $\overleftrightarrow{AB}$ makes two angles with the positive direction of the x-axis.

- One of them is positive (**i.e.** It has a positive measure) taken from the positive direction of the x-axis to the straight line in the direction of anticlockwise and it is $\angle$ DCA

- The another one is negative (i.e. It has a negative measure) taken from the positive direction of the X-axis to the straight line in the direction of clockwise and it is $\angle$ DCB



The slope of the straight line

Definition

The slope of the straight line is the tangent of the positive angle which this straight line makes with the positive direction of the X-axis.

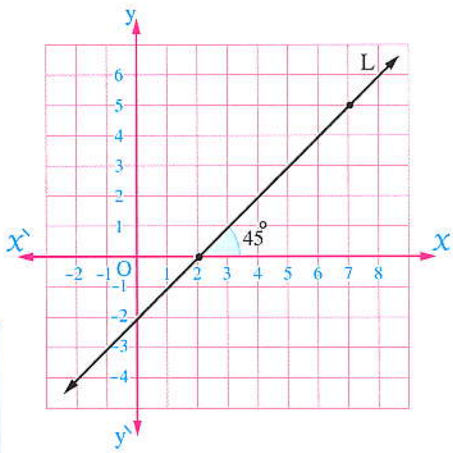
i.e. The slope of the straight line = $\tan \theta$ where θ is the measure of the positive angle which the straight line makes with the positive direction of the X-axis.

For example:

In the opposite figure :

The straight line L makes an angle of measure 45° with the positive direction of the X-axis, then :

the slope of the straight line L = $\tan 45° = 1$



Notice that :

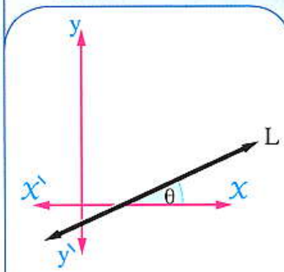
The straight line passes through the two points (2 , 0) and (7 , 5) , then : the slope of the straight line

$$L = \frac{y_2 - y_1}{X_2 - X_1} = \frac{5-0}{7-2} = \frac{5}{5} = 1$$

! Remark

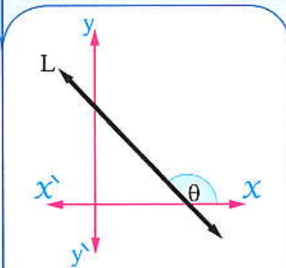
The angle which the straight line L makes with the positive direction of the X-axis takes one of the following cases :

(1) Acute angle



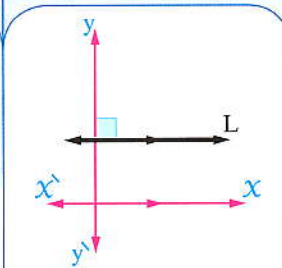
The slope is positive

(2) Obtuse angle



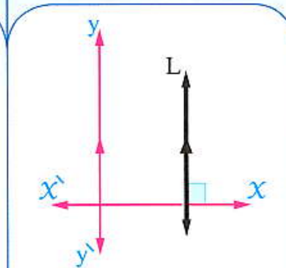
The slope is negative

(3) Zero angle



The slope is zero

(4) Right angle



The slope is undefined

Example 1

Find the slope of the straight line which makes a positive angle with the positive direction of X-axis where the measure of the angle is :

1 45° **2** $124^\circ\ 15'\ 12''$

Solution

1 The slope of the straight line $= \tan 45^\circ = 1$ Start → tan 4 5 =

2 The slope of the straight line $= \tan 124^\circ\ 15'\ 12'' \approx -1.4685$

Start → tan 1 2 4 °′″ 1 5 °′″ 1 2 °′″ =

Example 2

Find the measure of the positive angle (θ) which the straight line makes with the positive direction of X-axis if the slope of the straight line is :

1 1.486 **2** $-\frac{1}{\sqrt{3}}$

Solution

1 $\because m = \tan\theta$ $\therefore \tan\theta = 1.486$

$\because$ The slope is positive $\therefore \angle\theta$ is an acute angle.

Start → SHIFT tan 1 . 4 8 6 = °′″

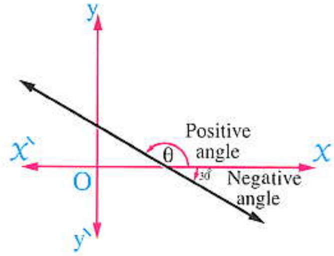
$\therefore m(\angle\theta) \approx 56^\circ\ 3'\ 41''$

2 $\because m = \tan\theta$ $\therefore \tan\theta = -\frac{1}{\sqrt{3}}$

, $\because$ the slope is negative $\therefore \angle\theta$ is an obtuse angle.

By using the calculator as follows :

Start → SHIFT tan (−) 1 ÷ √ 3 =



We will find the calculator gives the result -30°

Where the calculator is programmed to get the acute angle only either negative or positive.

But the required is the positive angle , so we find $m(\angle\theta)$ by finding the supplementary of the angle of measure 30°

Then : $m(\angle\theta) = 180^\circ - 30^\circ = 150^\circ$

Example 3

Find the measure of the positive angle (θ) which the straight line L makes with the positive direction of X-axis if the straight line (L) passes through the two points :

1 $\left(-2, \sqrt{3}\right), \left(1, 4\sqrt{3}\right)$ **2** $(-2, 3), (-3, 4)$

Solution

1 ∵ The straight line L passes through the two points $\left(-2, \sqrt{3}\right), \left(1, 4\sqrt{3}\right)$

∴ The slope of the straight line L

$$= \frac{4\sqrt{3}-\sqrt{3}}{1-(-2)} = \frac{3\sqrt{3}}{3} = \sqrt{3}$$

Notice that : The slope is positive, then the angle is acute.

Start → SHIFT tan √ 3 =

∴ m (∠ θ) = 60°

2 ∵ The straight line passes through the two points (– 2, 3) and (– 3, 4)

∴ The slope of the straight line L

$$= \frac{4-3}{-3-(-2)} = -1$$

Notice that : The slope is negative, then the angle is obtuse.

By using the calculator as follows :

Start → SHIFT tan (–) 1 =

We will find that, the calculator gives the result – 45° (a negative acute angle)

We will find the positive obtuse angle as follows :

m (∠ θ) = 180° – 45° = 135°

TRY by yourself 1

1 Find the slope of the straight line which makes a positive angle with the positive direction of X-axis with measure :

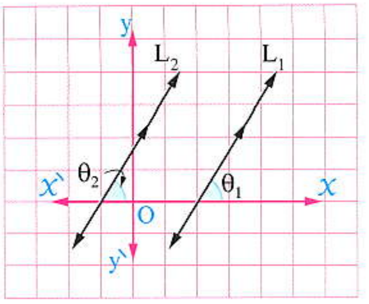
(1) 30° (2) 54° 30′ 6″ (3) 120°

2 Find the measure of the positive angle which the straight line makes with the positive direction of X-axis if the slope of the straight line = 6.2

3 Find the measure of the positive angle (θ) which the straight line L makes with the positive direction of X-axis if the straight line L passes through the two points (4, –1) and (5, –3)

The relation between the two slopes of two parallel straight lines

In the opposite figure :



If L_1 and L_2 are two parallel straight lines of slopes m_1 and m_2 respectively and make two positive angles with the positive direction of X-axis of measures θ_1 and θ_2 respectively , then

$\because L_1 // L_2$ $\therefore \theta_1 = \theta_2$ corresponding angles

$\therefore \tan\theta_1 = \tan\theta_2$ $\therefore m_1 = m_2$

thus we deduce the following :

If $L_1 // L_2$, then $m_1 = m_2$

i.e. If two straight lines are parallel , then their slopes are equal.

Also , we can deduce the opposite :

If $m_1 = m_2$, then $L_1 // L_2$

i.e. If the two straight lines have equal slopes , then the two straight lines are parallel.

Example 4

Prove that : The straight line which passes through the two points (2 , 3) and (–1 , 6) is parallel to the straight line which makes with the positive direction of X-axis a positive angle of measure 135°

Solution

The slope of the first straight line $m_1 = \frac{6-3}{-1-2} = \frac{3}{-3} = -1$

, the slope of the second straight line $m_2 = \tan 135° = -1$

$\therefore m_1 = m_2$ $\therefore$ The two straight lines are parallel.

Example 5

If A (–1 , 2) , B (2 , 3) , C (– 4 , 1) and D (X , 2) are four points in the Cartesian coordinates plane and $\overleftrightarrow{AB} // \overleftrightarrow{CD}$, **find the value of :** X

Solution

$\because \overleftrightarrow{AB} // \overleftrightarrow{CD}$

$\therefore$ The slope of the straight line passes through A (– 1 , 2) and B (2 , 3) is equal to the slope of the straight line passes through C (– 4 , 1) and D (X , 2)

$\therefore \frac{3-2}{2-(-1)} = \frac{2-1}{X-(-4)}$ $\therefore \frac{1}{3} = \frac{1}{X+4}$

$\therefore X + 4 = 3$ $\therefore X = -1$

Example 6 In the Cartesian coordinates plane , prove that the points A (–1 , 6) , B (3 , – 4) and C (2 , –1.5) are collinear.

Solution $\because$ The slope of $\overleftrightarrow{AB} = \frac{-4-6}{3-(-1)} = \frac{-10}{4} = -\frac{5}{2}$

, the slope of $\overleftrightarrow{BC} = \frac{-1.5-(-4)}{2-3} = \frac{2.5}{-1} = -2\frac{1}{2} = -\frac{5}{2}$

$\therefore$ The slope of $\overleftrightarrow{AB}$ = the slope of $\overleftrightarrow{BC}$ $\quad \therefore \overleftrightarrow{AB} \parallel \overleftrightarrow{BC}$

, $\because$ B is a common point between $\overleftrightarrow{AB}$ and $\overleftrightarrow{BC}$

$\therefore$ A , B and C are collinear.

Notice that :
If the slope of $\overleftrightarrow{AB}$ = the slope of $\overleftrightarrow{BC}$, then A , B and C are collinear points.

TRY by yourself 2

1. ***Prove that :*** The straight line L_1 passing through the two points (1 , 5) and (– 2 , – 1) is parallel to the straight line L_2 that passes through the two points (0 , – 1) and (5 , 9)

2. If the straight line $\overleftrightarrow{AB}$ // the X-axis where A (5 , – 4) and B (– 2 , y) , ***find the value of :*** y

The relation between the two slopes of two perpendicular (orthogonal) straight lines

If L_1 and L_2 are two straight lines of slopes m_1 and m_2 respectively and $L_1 \perp L_2$, then $m_1 \times m_2 = -1$, unless one of them is parallel to one of the coordinate axes.

i.e. The product of the slopes of the perpendicular straight lines = – 1

and vice versa : If L_1 and L_2 are two straight lines of slopes m_1 and m_2 respectively and $m_1 \times m_2 = -1$, then $L_1 \perp L_2$

i.e. If the product of the two slopes of two straight lines equals –1 , then the two straight lines are perpendicular (orthogonal)

Example 7 **Prove that :** The straight line L_1 which passes through the two points (–1 , 4) and (3 , 7) is perpendicular to the straight line L_2 which passes through the two points (1 , 1) and (4 , –3)

Solution $\because$ The slope of $L_1 = \frac{7-4}{3-(-1)} = \frac{3}{4}$, the slope of $L_2 = \frac{-3-1}{4-1} = -\frac{4}{3}$

, $\because$ the slope of L_1 × the slope of $L_2 = \frac{3}{4} \times -\frac{4}{3} = -1$ $\qquad \therefore L_1 \perp L_2$

Example 8 In the Cartesian coordinates plane, if the points A (1 , 7) , B (2 , 4) and C (5 , y) represent the vertices of a right-angled triangle at B ,

find the value of : y

Solution

$\because$ The slope of $\overleftrightarrow{AB} = \frac{4-7}{2-1} = -3$, the slope of $\overleftrightarrow{BC} = \frac{y-4}{5-2} = \frac{y-4}{3}$,

$\because \overleftrightarrow{AB} \perp \overleftrightarrow{BC}$ $\therefore$ The slope of $\overleftrightarrow{AB}$ × the slope of $\overleftrightarrow{BC} = -1$

$\therefore -3 \times \frac{y-4}{3} = -1$ $\therefore y - 4 = 1$ $\therefore y = 5$

! Remark

If $L_1 \perp L_2$, the slope of L_1 is m_1 and the slope of L_2 is m_2 where $m_1 \in \mathbb{R}^*$, $m_2 \in \mathbb{R}^*$, then $m_2 = \frac{-1}{m_1}$, $m_1 = \frac{-1}{m_2}$

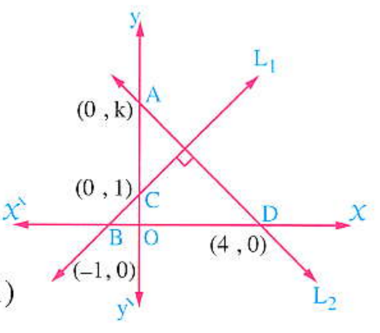
For example:

- If the slope of the straight line L is 2 , then the slope of the perpendicular to it $= -\frac{1}{2}$
- If the slope of the straight line L is $-\frac{2}{3}$, then the slope of the perpendicular to it $= \frac{3}{2}$

Example 9 **In the opposite figure :**

If $L_1 \perp L_2$

Find : The value of k



Solution

$\because$ The straight line L_1 passes through the two points B (− 1 , 0) and C (0 , 1)

$\therefore$ The slope of $L_1 = \frac{1-0}{0-(-1)} = 1$

, $\because$ the straight line L_2 passes through the two points A (0 , k) and D (4 , 0)

$\therefore$ The slope of $L_2 = \frac{0-k}{4-0} = -\frac{k}{4}$ (1)

, $\because L_1 \perp L_2$, the slope of $L_1 = 1$ $\therefore$ The slope of $L_2 = -1$ (2)

From (1) and (2) : $\therefore -\frac{k}{4} = -1$ $\therefore k = 4$

TRY by yourself 3

1. If A (− 2 , 5) , B (1 , 2) and C (3 , 4) are three points in a Cartesian coordinates plane , ***prove that :*** $\overleftrightarrow{AB} \perp \overleftrightarrow{BC}$
2. ***Prove that :*** The straight line which passes through the two points (7 , − 1) and (5 , −3) is perpendicular to the straight line which makes with the positive direction of x-axis an angle of measure 135°

Remarks to solve the problems on quadrilateral

To prove that a quadrilateral is a trapezium , we prove that :

Two opposite sides are parallel and the other two sides are not parallel.

To prove that a quadrilateral is a parallelogram , we prove only one of the following properties :

1. Each two opposite sides are parallel.
2. Each two opposite sides are equal in length.
3. Two opposite sides are parallel and equal in length.
4. The two diagonals bisect each other.

To prove that a quadrilateral is a rectangle, rhombus or square, we prove at first that the quadrilateral is a parallelogram, then :

- **To prove that the parallelogram is a rectangle, we prove only one of the following two properties :**

  (1) Two adjacent sides are perpendicular.

  (2) The two diagonals are equal in length.

- **To prove that the parallelogram is a rhombus, we prove only one of the following two properties :**

  (1) Two adjacent sides are equal in length.

  (2) The two diagonals are perpendicular.

- **To prove that the parallelogram is a square, we prove only one of the following properties :**

  (1) Two adjacent sides are perpendicular and equal in length.

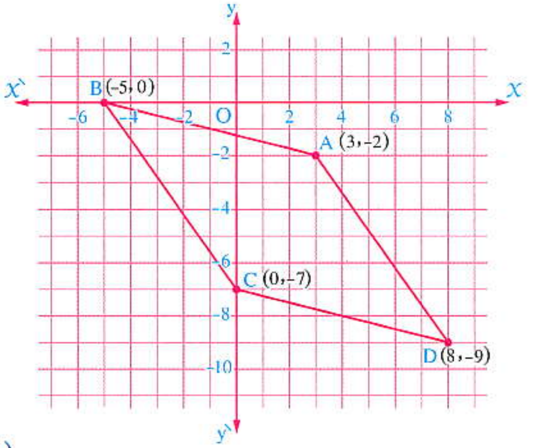
  (2) Two adjacent sides are perpendicular and its diagonals are perpendicular.

  (3) Two diagonals are equal in length and perpendicular.

  (4) Two adjacent sides are equal in length and its two diagonals are equal in length.

Example 10 On a Cartesian coordinates plane , represent the points A (3 , – 2) , B (– 5 , 0) , C (0 , – 7) and D (8 , – 9) , then prove that the quadrilateral ABCD is a parallelogram.

Solution



$\because$ The slope of $\overleftrightarrow{AB} = \frac{0-(-2)}{-5-3} = \frac{2}{-8} = -\frac{1}{4}$

, the slope of $\overleftrightarrow{CD} = \frac{-9-(-7)}{8-0} = \frac{-2}{8} = -\frac{1}{4}$

$\therefore$ The slope of $\overleftrightarrow{AB}$ = the slope of $\overleftrightarrow{CD}$

$\therefore \overleftrightarrow{AB} \,//\, \overleftrightarrow{CD}$ (1)

$\because$ The slope of $\overleftrightarrow{AD} = \frac{-9-(-2)}{8-3} = \frac{-7}{5}$, the slope of $\overleftrightarrow{BC} = \frac{-7-0}{0-(-5)} = \frac{-7}{5}$

$\therefore$ The slope of $\overleftrightarrow{AD}$ = the slope of $\overleftrightarrow{BC}$

$\therefore \overleftrightarrow{AD} \,//\, \overleftrightarrow{BC}$ (2)

From (1) and (2) : $\therefore$ The quadrilateral ABCD is a parallelogram.

Example 11 **Prove that :** The points A (2 , – 2) , B (8 , 4) , C (5 , 7) and D (–1 , 1) are vertices of the rectangle ABCD

Solution

$\because$ The slope of $\overleftrightarrow{AB} = \frac{-2-4}{2-8} = \frac{-6}{-6} = 1$

, the slope of $\overleftrightarrow{CD} = \frac{7-1}{5-(-1)} = \frac{6}{6} = 1$

$\therefore$ The slope of $\overleftrightarrow{AB}$ = the slope of $\overleftrightarrow{CD}$ $\therefore \overleftrightarrow{AB} \,//\, \overleftrightarrow{CD}$ (1)

$\because$ The slope of $\overleftrightarrow{AD} = \frac{-2-1}{2-(-1)} = \frac{-3}{3} = -1$

, the slope of $\overleftrightarrow{BC} = \frac{4-7}{8-5} = \frac{-3}{3} = -1$

$\therefore$ The slope of $\overleftrightarrow{AD}$ = the slope of $\overleftrightarrow{BC}$ $\therefore \overleftrightarrow{AD} \,//\, \overleftrightarrow{BC}$ (2)

From (1) and (2) we deduce that the quadrilateral ABCD is a parallelogram.

$\because$ The slope of $\overleftrightarrow{AB}$ × the slope of $\overleftrightarrow{BC} = 1 \times -1 = -1$

$\therefore \overleftrightarrow{AB} \perp \overleftrightarrow{BC}$ $\therefore$ The quadrilateral ABCD is a rectangle.

Example 12 On a Cartesian coordinates plane , represent the points A (– 3 , – 3) , B (3 , 1) , C (1 , 5) and D (– 2 , 3) , then prove that the quadrilateral ABCD is a trapezium.

Solution

$\because$ The slope of $\overleftrightarrow{CD} = \frac{5-3}{1-(-2)} = \frac{2}{3}$

, the slope of $\overleftrightarrow{AB} = \frac{1-(-3)}{3-(-3)} = \frac{4}{6} = \frac{2}{3}$

$\therefore$ The slope of $\overleftrightarrow{CD}$ = the slope of $\overleftrightarrow{AB}$

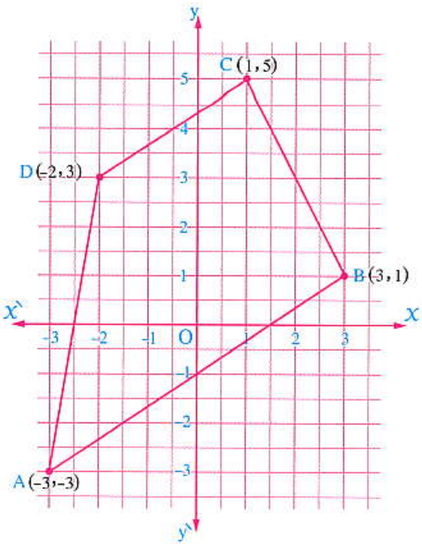
$\therefore \overleftrightarrow{CD} \parallel \overleftrightarrow{AB}$ (1)

The slope of $\overleftrightarrow{BC} = \frac{5-1}{1-3} = -2$

, the slope of $\overleftrightarrow{AD} = \frac{3-(-3)}{-2-(-3)} = 6$

$\therefore$ The slope of $\overleftrightarrow{BC} \neq$ the slope of $\overleftrightarrow{AD}$

$\therefore \overleftrightarrow{BC}$ is not parallel to $\overleftrightarrow{AD}$ (2)



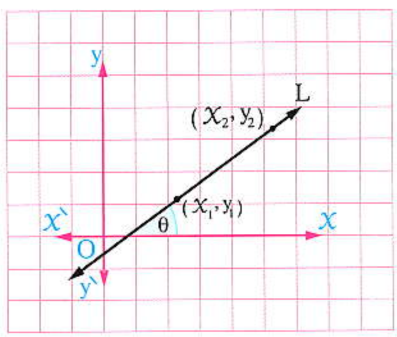
From (1) and (2) :

$\therefore$ The quadrilateral ABCD is a trapezium.

Lesson 4

The equation of the straight line given its slope and the intercepted part of y-axis

We studied before that the relation : $a\,X + b\,y + c = 0$ where $a \neq 0$, $b \neq 0$ together is a linear relation represented graphically by a straight line and we can find its slope (m) by one of the following methods :



1 $m = \dfrac{\text{vertical change}}{\text{horizontal change}} = \dfrac{y_2 - y_1}{X_2 - X_1}$

Where (X_1 , y_1) and (X_2 , y_2) are two points on the straight line

2 $m = \tan\theta$

Where θ is the measure of the positive angle which the straight line makes with the positive direction of the X-axis.

- **We will continue our study about this subject by studying how :**
  - To find the slope of the straight line and the length of the intercepted part from y-axis if we know the equation of the straight line.
  - To find the equation of the straight line if we know its slope and the length of the intercepted part from the y-axis.

First — Finding the slope of the straight line and the length of the intercepted part of y-axis

Prelude example

Represent graphically the relation : $2x - y + 3 = 0$ and from the graph , find the slope of the straight line which represents the relation and the intercepted part of the y-axis by the straight line.

Solution

To graph the straight line which represents the relation , find two points of the points of the straight line at least , to facilitate that , put one of the variables x or y in a side of the equation

$\because 2x - y + 3 = 0 \qquad \therefore -y = -2x - 3$

$\therefore y = 2x + 3$

At $x = 0 \qquad \therefore y = 0 + 3 = 3$

$\therefore (0 , 3)$ is one of the points of the straight line.

At $x = -1 \qquad \therefore y = -2 + 3 = 1$

$\therefore (-1 , 1)$ is one of the points of the straight line.

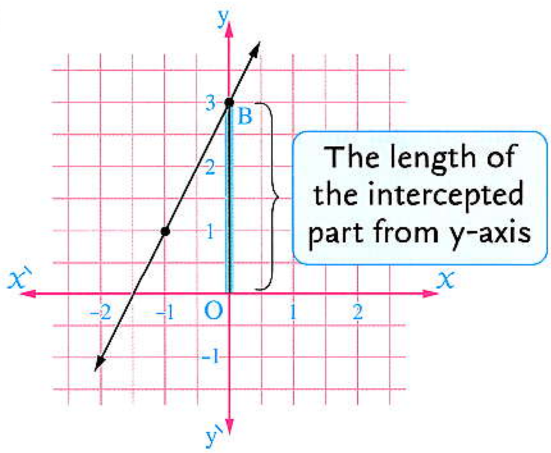
i.e. The straight line passes through the two points $(0 , 3)$ and $(-1 , 1)$

$$\therefore \text{The slope of the straight line} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1-3}{-1-0} = \frac{-2}{-1} = 2$$

- From the graph, we find that :

  OB = 3 length units.

i.e. The straight line intercepts from the positive part from y-axis 3 length units



Observing the graph of the straight line : $y = 2x + 3$

We find that :

- The slope of the straight line = the coefficient of $x = 2$
- The length of the intercepted part from y-axis = | absolute term | = | 3 | = 3 length units.

The slope of the straight line

$$y = \textcircled{2}\, x + \textcircled{3}$$

The length of the intercepted part from y-axis i.e.

If the equation of a straight line is in the form : $y = m\mathcal{X} + c$, then :

- The slope of the straight line = m
- The length of the intercepted part from y-axis = | c |
  and it passes through the point (0 , c)

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Example 1 Find the slope of the straight line : $2\mathcal{X} + 5y - 15 = 0$, then find the intercepted part of y-axis.

Solution Write the equation of the straight line in the form : $y = m\mathcal{X} + c$

$\therefore 5y = -2\mathcal{X} + 15$ $\qquad \therefore y = \frac{-2}{5}\mathcal{X} + 3$

$\therefore$ The slope of the straight line $= \frac{-2}{5}$ and the intercepted part of the positive part of y-axis is of length = 3 length units.

! Remark

In the previous example , observing the equation in the form : $2\mathcal{X} + 5y - 15 = 0$, we find that :

- The slope of the straight line $= \frac{-\text{coefficient of } \mathcal{X}}{\text{coefficient of } y} = \frac{-2}{5}$
- The straight line cuts y-axis at the point $\left(0 , \frac{-\text{absolute term}}{\text{coefficient of } y}\right)$ i.e. (0 , 3)

i.e. The straight line intercepts a part of y-axis of length $= \left|\frac{-\text{absolute term}}{\text{coefficient of } y}\right|$

$= |3| = 3$ length units.

i.e.

If the equation of a straight line is in the form : $a\mathcal{X} + by + c = 0$, then

- The slope of the straight line $= \frac{-\text{coefficient of } \mathcal{X}}{\text{coefficient of } y} = \frac{-a}{b}$
- The straight line cuts y-axis at the point $\left(0 , \frac{-c}{b}\right)$

i.e. The length of the intercepted part from y-axis $= \left|\frac{-c}{b}\right|$

For example:

- The straight line whose equation is : $x - 2y + 3 = 0$

  Its slope $= \frac{-1}{-2} = \frac{1}{2}$ and cuts y-axis at the point $\left(0 , \frac{3}{2}\right)$

  i.e. The straight line intercepts a part of length $\frac{3}{2}$ length unit from the positive part of y-axis.

- The straight line whose equation is : $3x + y + 4 = 0$

  Its slope $= -3$ and cuts y-axis at the point $(0 , -4)$

  i.e. The straight line intercepts a part of length 4 length units from the negative part of y-axis.

Example 2 If the straight line that passes through the two points $(-1 , 7)$ and $(9 , 3)$ is perpendicular to the straight line whose equation is : $x + ky - 13 = 0$, **find the value of :** k

Solution Let the slope of the straight line that passes through the two points $(-1 , 7)$ and $(9 , 3)$ be m_1

$$\therefore m_1 = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3 - 7}{9 - (-1)} = \frac{-4}{10} = \frac{-2}{5}$$

Let the slope of the straight line whose equation is : $x + ky - 13 = 0$ be m_2

$$\therefore m_2 = \frac{-a}{b} = \frac{-1}{k}$$

$\because$ The two straight lines are perpendicular

$\therefore m_1 \times m_2 = -1$ $\qquad \therefore \frac{-2}{5} \times \frac{-1}{k} = -1$

$\therefore \frac{2}{5k} = -1$ $\qquad \therefore -5k = 2$ $\qquad \therefore k = \frac{-2}{5}$

TRY by yourself 1

1. If the two straight lines : $3y + x - 7 = 0$ and $y = kx + 5$ are perpendicular , then find the value of : k
2. Find the measure of the positive angle which is made by the straight line whose equation is : $3x - 3y + 5 = 0$ with the positive direction of x-axis.
3. Find the length of the intercepted part from y-axis by the straight line whose equation is : $2y = 3x + 12$

Second: Finding the equation of the straight line given its slope and the length of intercepted part of y-axis

The straight line whose slope = m and cuts y-axis at the point (0 , c) its equation is in the form :

$$y = m x + c$$

Example 3 **Find the equation of the straight line :**

1. Whose slope $= -\frac{3}{4}$ and intercepts from the positive part of y-axis 3 length units.
2. Whose slope = 2 and intercepts from the negative part of y-axis 7 length units.

Solution $y = m x + c$

1. $\because m = -\frac{3}{4}$, $c = 3$ $\quad\therefore$ The equation is : $y = -\frac{3}{4} x + 3$
2. $\because m = 2$, $c = -7$ $\quad\therefore$ The equation is : $y = 2 x - 7$

Example 4 Find the equation of the straight line which makes with the positive direction of x-axis a positive angle of measure 135° and intercepts from the positive part of y-axis a part of length 7 length units.

Solution $\because$ The slope = tan 135° = – 1

$\therefore$ The equation of the straight line is : $y = -x + 7$

! Remarks

1. The equation of the straight line which passes through the origin point O (0 , 0) is $y = m x$, where m is the slope of the straight line.
2. The equation of x-axis is $y = 0$
3. The equation of y-axis is $x = 0$
4. The equation of the straight line which is parallel to x-axis and passes through the point $(0 , \ell)$ is $y = \ell$
5. The equation of the straight line which is parallel to y-axis and passes through the point (k , 0) is $x = k$

Example 5 Find the equation of the straight line which passes through the two points (1 , – 1) and (2 , 2)

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Solution Let the equation of the straight line be in the form $y = m x + c$

$\because$ The slope (m) $= \frac{y_2 - y_1}{x_2 - x_1} = \frac{2-(-1)}{2-1} = 3$

$\therefore$ The equation of the straight line is in the form : $y = 3x + c$

$\because$ (1 , – 1) belongs to the straight line.

$\therefore -1 = 3 \times 1 + c$ $\qquad$ $\therefore c = -1 - 3 = -4$

$\therefore$ The equation of the straight line is : $y = 3x - 4$

Example 6 Find the equation of the straight line which passes through the point (1 , 2) and parallels the straight line $2x + 3y - 6 = 0$

Solution $\because$ The slope of the given straight line $= \frac{-\text{coefficient of } x}{\text{coefficient of } y} = \frac{-2}{3}$

$\therefore$ The slope of the required straight line $= \frac{-2}{3}$

$\therefore$ The equation of the required straight line is : $y = -\frac{2}{3}x + c$

$\because$ The straight line passes through the point (1 , 2)

$\therefore 2 = -\frac{2}{3} \times 1 + c$ $\qquad$ $\therefore c = \frac{8}{3}$

$\therefore$ The equation of the required straight line is : $y = -\frac{2}{3}x + \frac{8}{3}$

Example 7 Find the equation of the straight line which passes through the point (2 , 3) and perpendicular to the straight line passing through the two points A (3 , – 4) and B (5 , – 3)

Solution $\because$ The slope of the straight line which passes through the two points (3 , – 4) and (5 , – 3) equals $\frac{-3-(-4)}{5-3} = \frac{1}{2}$

$\therefore$ The slope of the required straight line $= -2$

$\therefore$ The equation of the required straight line is $y = -2x + c$

$\because$ The straight line passes through the point (2 , 3)

$\therefore$ It satisfies the equation.

$\therefore 3 = -2 \times 2 + c$ $\qquad$ $\therefore c = 7$

$\therefore$ The equation of the required straight line is : $y = -2x + 7$

TRY by yourself 2

1. Find the equation of the straight line which intercepts from the positive part of y-axis 5 length units and it is parallel to the straight line passing through the two points (– 2 , 3) and (– 1 , – 6)

2. Find the equation of the straight line which passes through the point (3 , 4) and perpendicular to $\overleftrightarrow{AB}$, where A (2 , – 3) and B (5 , 4)

Example 8 ABC is a triangle whose vertices are A (1 , 2) , B (– 2 , 3) , C (– 4 , – 3) , $\overline{AD}$ is a median of it , find the equation of $\overleftrightarrow{AD}$

Solution

$\because$ $\overline{AD}$ is a median of Δ ABC

$\therefore$ D is the midpoint of $\overline{BC}$

$\therefore D = \left(\frac{-2 + (-4)}{2} , \frac{3 + (-3)}{2} \right) = (-3 , 0)$

$\therefore$ The slope of $\overleftrightarrow{AD} = \frac{2 - 0}{1 - (-3)} = \frac{1}{2}$

$\therefore$ The equation of $\overleftrightarrow{AD}$ is : $y = \frac{1}{2} x + c$

$\because$ $\overleftrightarrow{AD}$ passes through the point A = (1 , 2)

$\therefore$ It satisfies its equation

$\therefore 2 = \frac{1}{2} \times 1 + c$ $\qquad$ $\therefore c = \frac{3}{2}$

$\therefore$ The equation of $\overleftrightarrow{AD}$ is : $y = \frac{1}{2} x + \frac{3}{2}$

TRY by yourself 3

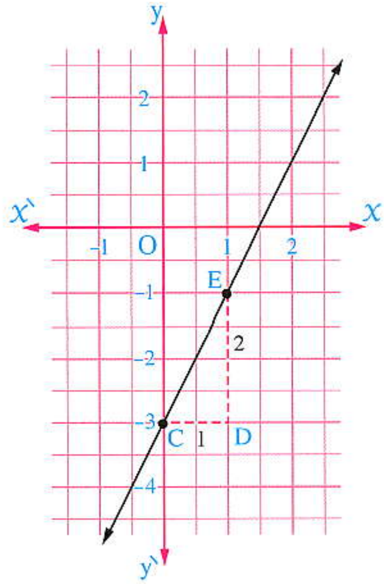
ABC is a triangle whose vertices are A (– 1 , 5) , B (4 , – 2) and C (– 3 , 0) Find the equation of the straight line passing through A and perpendicular to $\overline{BC}$

Example 9 Using the slope and the intercepted part of y-axis, represent graphically the straight line whose equation is $y = 2x - 3$

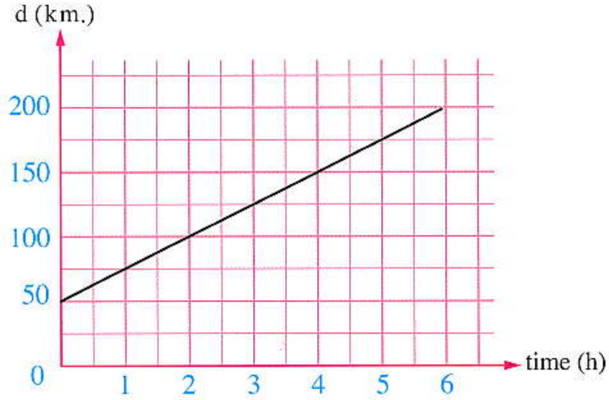
Solution The slope of the straight line $= 2 = \frac{2}{1} = \frac{\text{vertical change}}{\text{horizontal change}}$

and the straight line passes through the point C (0 , – 3)

From the point C , we move horizontally towards the right one unit (the horizontal change (+1)) to reach the point D , then we move vertically towards up two units (the vertical change (+2)) to reach the point E, then $\overleftrightarrow{CE}$ is the graph of the equation of the straight line $y = 2x - 3$



Example 10 The opposite graph represents the motion of a car moving with a uniform velocity where the distance (d) is measured in km. and the time (t) in hours , **find :**



1 The distance (d) at the beginning of the motion.

2 The velocity of the car.

3 The equation of the straight line representing the motion of the car.

Solution

1 The distance (d) at the beginning of the motion = 50 kilometres.

2 The velocity of the car = the slope of the straight line passing through the two points (0 , 50) and (6 , 200) $= \frac{200 - 50}{6 - 0} = \frac{150}{6}$

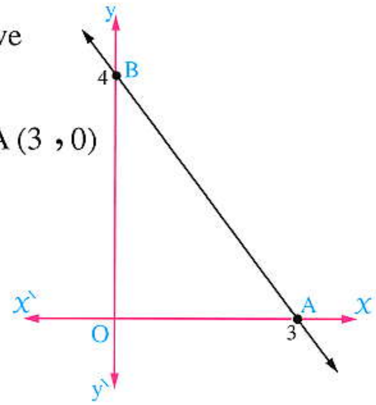
$= 25$ km./hr.

3 The equation of the straight line is : $d = m\,t + c$

i.e. $d = 25\,t + 50$

Example 11 Find the equation of the straight line which intercepts from the coordinate axes (X-axis and y-axis) two positive parts with lengths 3 and 4 length units respectively, then find the area of the triangle included between the straight line and the two axes.

Solution



$\because$ The straight line intercepts from the positive part of X-axis 3 length units.

$\therefore$ The straight line passes through the point A (3 , 0)

$\because$ The straight line intercepts from the positive part of y-axis 4 length units.

$\therefore$ The straight line passes through the point B (0 , 4)

$\therefore$ The straight line passes through the two points A (3 , 0) and B (0 , 4)

Let the equation of the required straight line be $y = mX + c$

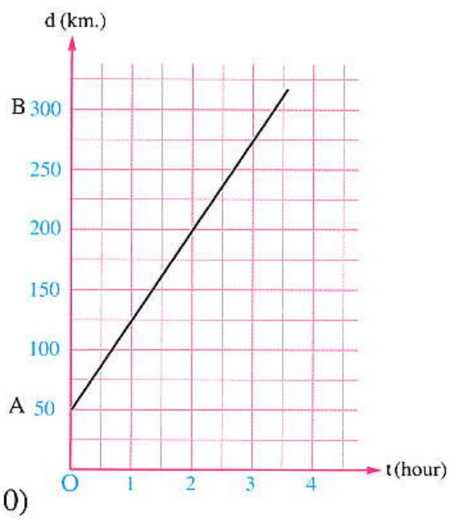
, where the slope (m) $= \frac{4-0}{0-3} = -\frac{4}{3}$ $\quad \therefore y = -\frac{4}{3}X + c$

, $\because c = 4$ $\quad \therefore$ The equation is : $y = -\frac{4}{3}X + 4$

, the area of Δ ABO $= \frac{1}{2} \times AO \times BO = \frac{1}{2} \times 3 \times 4 = 6$ square units.

TRY by yourself 4

A person moved between the cities A and B using his car with a uniform velocity and the opposite graph represents the relation between the distance (d) in kilometres and the time (t) in hours.



Answer the following :

1. What is the uniform velocity of the car ?
2. Find the equation of the straight line representing the motion of the car.
3. Find the distance between the car and O (0 , 0) after 3 hours from the beginning of the motion.

Notes

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